

Evaluation of a Photon Sensitivity Integral for the FAME Instrument

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Note: This document is actually a Maple session. Normal text is shown in Times Roman black. Maple input is in *Courier red* and, for most, can (perhaps *should*) be ignored. Comments embedded in Maple input begin with the pound symbol, #. Maple output is shown in *Times Roman blue* for math expressions and *Courier blue* for text (informational messages). Maple input commands are normally terminated with a semicolon; when an input command terminates with a full colon, output to the screen is suppressed. This is a handy mechanism for suppressing unwanted output, and I use it frequently. Hyperlinks allow you to jump around in the document; they appear as underlined blue. Here is a Table of Contents for this Memo:

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[-] Introduction

J.D. Phillips (SAO) is currently examining the photon sensitivity in the focal plane of the FAME instrument (SAO TM97-xx). Among other things, the calculation involves the Planck function (as an approximation of stellar spectra) convolved with the intensity envelope due to diffraction of starlight through the FAME starlight aperture. Therefore, integrals of the form

$$S(r, s, g, T, v_1, v_2) = \int_{v_1}^{v_2} \frac{1 \sin(s v)^2 (\sin(r v) - \sin(r g v))^2}{e^{\left(\frac{h v}{k T}\right)} - 1} dv$$

need to be evaluated in a *computationally efficient manner*. (Here, r , s , and g are instrument geometry parameters, and $v_2 - v_1$ is the frequency bandpass.) This integral form is unavailable in Gradshteyn and Ryzhik, and Maple is unable to evaluate it.

The inability of the Risch algorithm, which Maple implements fully, to solve this integral *proves* that it has *no closed-form solution in terms of elementary functions*. Therefore, we are forced to make an approximation for the Planck function. A simple power series seems appropriate and has been shown by Phillips (SAO TM97-xx) to be satisfactory for the current application:

$$\frac{1}{e^{\left(\frac{h v}{k T}\right)} - 1} = \sum_k a_k v^{(k-1)}$$

Inclusion of terms to fourth order in frequency v appears to be adequate. The question then becomes: can we evaluate the resulting integral and present the result in a useful form?

This memo addresses this evaluation issue. In [section 1](#) the integral is defined, and it is shown that a simple, brute-force approach is inadequate. In [section 2](#) the indefinite integral is solved and coerced into simplified form. A fortran subroutine is created that performs the integral evaluation in as computationally efficient a manner as is probably possible. In [section 3](#), I differentiate the integral and recover the integrand, lending some confidence in the integration result. We would like a more rigorous and independent check, however. So in [section 4](#) I solve the definite integral by successive application of integration by parts. After a lengthy calculation, I obtain a useful result, though it is not quite as computationally efficient as the indefinite integration result. In [section 5](#), I compare the definite and indefinite integrations and show that they are indeed equal.

This lends great confidence that the answer is correct. In [section 6](#) we consider the fact that the found solutions are invalid when the geometrical parameters s , r , and g lie on certain surfaces in (s, r, g) space. In general, series expansions across these singular surfaces are difficult to integrate. For the special case $s = r = 0$, I provide the series expansion that covers this region, an error function for deciding when to use the series expansion, and fortran subroutines for corresponding numerical calculations. Finally, in the [Appendix](#), I present for reference the full expressions of the indefinite and definite integrations, as well as the optimized fortran subroutine for the full integral

└ evaluation.

[-] 1. Definite Integral

Here is the integration kernel (the diffraction intensity envelope):

$$\text{integrand}_0 := \sin(s x)^2 (\sin(r x) - \sin(r g x))^2$$

Here is the whole integrand:

$$\text{integrand}_1 := \text{convert}([\text{seq}(a_k x^k, k = 0 .. 4)], +) \text{integrand}_0$$

$$\text{integrand}_1 := (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) \sin(s x)^2 (\sin(r x) - \sin(r g x))^2$$

Let's integrate just the kernel to get an idea of what we're up against.

$$\int_{v_1}^{v_2} \text{integrand}_0 dx$$

cost(%)

399 additions + 7156 multiplications + 26 divisions + 896 functions + 928 subscripts

Well now that's unpleasant. Even worse, attempting the full integral causes terminal constipation. Apparently, we'll have to approach the problem with brain engaged.

[-] 2. Indefinite Integral

Let's see what Maple will do with the *indefinite* integral.

$$S(r, s, g, T) = \int \text{integrand}_1 dx$$

$$S(r, s, g, T) = \int (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) \sin(s x)^2 (\sin(r x) - \sin(r g x))^2 dx$$

indef := value(%)

cost(rhs(%))

619 additions + 195 functions + 1761 multiplications + 70 divisions

That was easier than anticipated. We can do a bit more simplification of this expression.

indef := collect(*indef*, [sin, cos, seq(a_i, i = 0 .. 4), x], factor); cost(rhs(%))

437 additions + 26 functions + 959 multiplications + 240 divisions

Those 26 function calls are in fact all *unique* sines and cosines. To show this, isolate the *unique* trig terms and count them.

```
select(has, indets(indef), { sin, cos })  
  
{ sin(2 r g x), cos(2 r g x), sin(2 s x), cos(2 s x), sin((2 s + 2 r) x), cos((2 s + 2 r) x),  
sin((2 s - 2 r) x), cos((2 s - 2 r) x), sin((2 s + 2 r g) x), cos((2 s + 2 r g) x), sin(2 r x), cos(2 r x),  
cos((2 s - r + r g) x), sin((2 s - r + r g) x), cos((2 s + r - r g) x), sin((2 s + r - r g) x),  
sin((2 s + r + r g) x), cos((2 s + r + r g) x), sin((2 s - r - r g) x), cos((2 s - r - r g) x),  
sin((2 s - 2 r g) x), cos((2 s - 2 r g) x), sin((r + r g) x), cos((r + r g) x), sin((-r + r g) x),  
cos((-r + r g) x)}  
  
nops(%)
```

26

Write a procedure (it will also come in handy later) to collect terms on the sin/cos argument factors and thereby clean up the integration result even further. A fair amount of experimenting resulted in this procedure, the final form of which is shown below. Note that the existence of 26 unique trigonometric terms implies that, no matter how aggressive our simplification process, we are going to end up with a complicated result.

```
cleanup := proc( mess )  
  
local arglist, p, Set_sincos, candidate, tmp, Tlist, i;  
global time0;  
  
time0 := time();  
  
Set_sincos := select( has, indets(mess), {sin,cos} );  
debug_print(procname,`number of unique sin/cos terms:`,0,nops(%));  
  
# Extract the factors within the trig arguments.  
# Divide by 2 those that are exactly divisible by 2 and  
# throw those in, too.  
arglist := {}:  
for p in Set_sincos do  
    candidate := remove(has, op(1,p), {args[2..nargs]});  
    if gcd( candidate, 2 ) = 1 then  
        arglist := arglist union {candidate};  
    else  
        arglist := arglist union {candidate, candidate/2};  
    fi;  
od:  
arglist := convert(arglist,list);  
  
# Sort according to complexity.  
arglist := sort( arglist,  
    (a,b)->if length(a)>length(b) then true else false fi );  
debug_print(procname,`arg list:`,0,arglist);  
  
# Now replace each occurrence of an argument in the list  
# with a temporary, sort on trig and temporary-containing
```

```

# terms, then restore the temporaries.
debug_print(procname, `simplifying...`, 0);
tmp := mess:
Tlist := []:
for i from 1 to nops(arglist) do
    tmp := subs( arglist[i]=T.i, tmp );
    Tlist := [op(Tlist),T.i];
od;
tmp := collect(tmp,[sin,cos,op(Tlist)],factor);
for i from 1 to nops(arglist) do
    tmp := subs( T.i=arglist[i], tmp );
od;

debug_print( procname, `error check:`, 0,
            factor(expand(tmp-mess)) );

# We're done.
tmp;
end:

```

Okay, here we go.

```

cleanup(indef,x)
cleanup[0]: number of unique sin/cos terms:
26
cleanup[0]: arg list:
[2 s - r + r g, 2 s - r - r g, 2 s + r + r g, 2 s + r - r g, s - r g, 2 s - 2 r g, 2 s + 2 r g, -r + r g, r + r g,
 s + r g, 2 r g, 2 s - 2 r, s - r, 2 s + 2 r, s + r, r g, 2 s, 2 r, r, s]
cleanup[0]: simplifying...
cleanup[2]: error check:
0 = 0

```

*cost(rhs(%)); cost(rhs(*indef*))*

297 additions + 26 functions + 764 multiplications + 80 divisions

437 additions + 26 functions + 959 multiplications + 240 divisions

Our efforts have paid off.

indef:=%%%

Now we will create a Maple procedure out of this result. (A Maple procedure is needed in order to generate a fortran subroutine.) We first substitute for the a_i since the subscript notation will cause Maple problems in what comes next.

```

tmp := subs(seq(ai = a.i, i = 0 .. 4), rhs(indef))

readlib(optimize):
sensitivity_expr :=
optimize/makeproc([optimize(tmp, tryhard)], parameters = [x, seq(a.i, i = 0 .. 4), s, r, g])

```

The cost of the optimized expression sequence is

```

cost(optimize(tmp))
119 additions + 186 multiplications + 91 divisions + 26 functions + 108 assignments

```

This is about as simplified as we can possibly make this integration result. Finally, create a fortran subroutine from the Maple procedure and write it to disk.

```

fortran(sensitivity_expr, optimized, mode = double, precision = double,
filename = "d:/FAMEStuff/PhotonSensitivity/SensitivityIntegral.f")

```

For reference, both the indefinite integration result and the optimized fortran subroutine are shown in the [Appendix](#).

3. Check of the Indefinite Integral

Differentiate the indefinite integration result and check that it is equal to the integrand.

```

gawdawfulness := expand\left(\left(\frac{\partial}{\partial x} \text{rhs}(indef)\right) - \text{integrand}_1\right)

```

```
cost(%)

```

5770 additions + 27247 multiplications + 2328 divisions + 5754 functions + 2243 subscripts

```

collect(gawdawfulness, [sin, cos, seq(a_k, k = 0 .. 4), x], factor)

```

```
cost(%)

```

54 additions + 174 multiplications + 13 functions + 55 subscripts

```

factor(combine(%))

```

0

Whew.

4. Definite Integration by Parts

$$S(r, s, g, T, v_1, v_2) = \int_{v_1}^{v_2} \text{integrand}_1 dx$$

$$S(r, s, g, T, v_1, v_2) = \int_{v_1}^{v_2} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) \sin(s x)^2 (\sin(r x) - \sin(r g x))^2 dx$$

```

integral := rhs(%)

```

To consolidate effort, we now write a procedure that automatically separates out the trig and polynomial parts of the integrand and performs integration by parts with the polynomial term chosen as the one that is differentiated. Comments in the procedure body explain what's happening. This procedure is highly optimized for this particular form of integral.

```

ugh := proc( expr )
  local p, q, H, K, locI, locp, T, remdr;
  global parts, _k_;

  # update the parts index
  if not assigned(_k_) then
    _k_ := 0;
  else
    _k_ := _k_ + 1;
  fi;

  # grab the integral
  if type( expr, function ) and op(0,expr)=Int then
    H := expr;
  else
    H := select( has, expr, Int );
  fi;
  # get the integrand
  K := op(1,H);
  # grab the polynomial part and store in p
  q := remove( has, %, {sin,cos} );
  if type(q,`+`) then #already have it
    p := q;
  else
    for p in q do
      if type(p,`+`) then
        break;
      fi;
    od:
  fi;
  if not has(p,{s,g}) then
    locp := location( K, p );
    debug_print(procname,'polynomial factor p:`,1,p);
    debug_print(procname,'location of p in integrand: `,2,locp);
  else
    locp := [];
    debug_print(procname,'no polynomial factor!`,1);
  fi;

  # now integrate by parts
  debug_print(procname,'integrating by parts...',0);
  if nops(locp) > 0 then
    q := intparts( expr, p );
    debug_print(procname,'entire integrated mess:`,4,q);
  else
    debug_print(procname,'entire integrated mess:`,4);
    RETURN( value( expr ) );
  fi;

  # store the evaluated part
  parts[_k_] := remove( has, q, Int );
  debug_print( procname,

```

```

`cost of evaluated part of integral:`, 1,
cost(parts[_k_]) );
if _k_ > 0 then
  debug_print( procname, `simplifying...`, 1 );
  parts[_k_] := collect(parts[_k_], [sin,cos], factor);
  debug_print( procname,
    `cost of evaluated part of integral:`, 1,
    cost(parts[_k_]) );
fi;
debug_print( procname,
  `evaluated part of integral is stored in parts[`._.k_.`]`, 0 );

# select the remainder integral
remdr := select( has, q, Int );
debug_print(procname,`new integral expression:`,3,remdr);

# now we want to simplify the new (and messy) integrand

# grab the integral and store in H
if type( remdr, function ) and op(0,remdr)=Int then
  H   := remdr;
  locI := [];
else
  H   := select( has, remdr, Int );
  locI := location( remdr, H );
fi;
debug_print(procname,
  `location of integral in integral expression: `,3,locI);

# get the integrand and store in K
K := op(1,H);
debug_print(procname,`full integrand:`,3,K);

# grab the new polynomial part and store in p
q := remove( has, %, {sin,cos} );
if type(q,`+`) then
  p := q;
else
  for p in q do
    if type(p,`+`) then
      break;
    fi;
  od;
fi;
if not has(p,{s,g}) then
  locp := location( K, p );
  debug_print(procname,`new polynomial factor p:`,1,p);
  debug_print(procname,`location of p in integrand: `,2,locp);
  debug_print(procname,`full non-sin/cos term: `,1,q);
else
  locp := [];
  debug_print(procname,`no new polynomial factor!`,1);
fi;

# sub for the new polynomial part of the
# integrand, simplify the trig part, then sub back
debug_print(procname,`cost of new integrand:`,1,cost(K));
debug_print( procname,
  `simplifying sin/cos part of new integrand...`, 0 );

```

```

if nops(locp) > 0 then
    subsop( locp=T, K );
    collect( %, [T,cos,sin], factor );
    K := subs( T=p, % );
else
    K := collect( K, [cos,sin], factor );
fi;
debug_print( procname,
    `cost of simplified new integrand:`, 1, cost(K) );
# put back into integral and the integral back
# into the remainder integral expression and we're done
debug_print(procname,`new definite integral:`,0);
subsop( l=K, H );
subsop( locI=%, remdr );
end:

```

Now we begin the calculation. We'll apply this integration by parts procedure several times, until the polynomial term of the integrand is differentiated into oblivion. Notice the *huge* amount of simplification being done, as evidenced by the "cost" diagnostics.

```

time0 := time(): #reset print timing
_k_ := '_k_': #reset parts index
verbosity := 2: #not too verbose, please
ugh( integral );
ugh[0]: polynomial factor p:

```

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

```
ugh[0]: location of p in integrand:
```

```
[1]
```

```
ugh[0]: integrating by parts...
```

```
ugh[0]: cost of evaluated part of integral:
```

91 additions + 164 multiplications + 28 divisions + 26 functions + 46 subscripts

```
ugh[0]: evaluated part of integral is stored in parts[0]
```

```
ugh[0]: new polynomial factor p:
```

$$2 x a_2 + 3 x^2 a_3 + a_1 + 4 x^3 a_4$$

```
ugh[0]: location of p in integrand:
```

```
[1]
```

```
ugh[0]: full non-sin/cos term:
```

$$2 x a_2 + 3 x^2 a_3 + a_1 + 4 x^3 a_4$$

```
ugh[0]: cost of new integrand:
```

44 additions + 81 multiplications + 14 divisions + 13 functions + 4 subscripts

```
ugh[0]: simplifying sin/cos part of new integrand...
```

```
ugh[0]: cost of simplified new integrand:
```

44 additions + 71 multiplications + 16 divisions + 13 functions + 4 subscripts

```
ugh[0]: new definite integral:
```

$$\begin{aligned}
& - \int_{v_1}^{v_2} \left(-\frac{1}{8} \frac{\sin(2r g x)}{r g} - \frac{1}{4} \frac{\sin(2s x)}{s} + \frac{1}{16} \frac{\sin((2s+2r)x)}{s+r} + \frac{1}{16} \frac{\sin((2s-2r)x)}{s-r} \right. \\
& \quad + \frac{1}{16} \frac{\sin((2s+2rg)x)}{s+rg} - \frac{1}{8} \frac{\sin(2rx)}{r} + \frac{1}{4} \frac{\sin((2s-r+rg)x)}{2s-r+rg} + \frac{1}{4} \frac{\sin((2s+r-rg)x)}{2s+r-rg} \\
& \quad - \frac{1}{4} \frac{\sin((2s+r+rg)x)}{2s+r+rg} - \frac{1}{4} \frac{\sin((2s-r-rg)x)}{2s-r-rg} + \frac{1}{16} \frac{\sin((2s-2rg)x)}{s-r g} + \frac{1}{2} \frac{\sin((r+rg)x)}{r(1+g)} \\
& \quad \left. - \frac{1}{2} \frac{\sin((-r+rg)x)}{r(-1+g)} + \frac{1}{2} x \right) (2x a_2 + 3x^2 a_3 + a_1 + 4x^3 a_4) dx
\end{aligned}$$

ugh(%)

ugh[0]: polynomial factor p:

$$2x a_2 + 3x^2 a_3 + a_1 + 4x^3 a_4$$

ugh[0]: location of p in integrand:

[2]

ugh[0]: integrating by parts...

ugh[2]: cost of evaluated part of integral:

2697 additions + 13778 multiplications + 690 divisions + 728 functions + 774 subscripts

ugh[2]: simplifying...

ugh[6]: cost of evaluated part of integral:

174 additions + 466 multiplications + 32 divisions + 26 functions + 244 subscripts

ugh[6]: evaluated part of integral is stored in parts[1]

ugh[6]: new polynomial factor p:

$$2a_2 + 6xa_3 + 12x^2a_4$$

ugh[6]: location of p in integrand:

[2]

ugh[6]: full non-sin/cos term:

$$-\frac{1}{16}(2a_2 + 6xa_3 + 12x^2a_4)/(r g s (s+r)(s-r)(s+rg)(2s-r+rg)(2s+r-rg))$$

$$(2s+r+rg)(2s-r-rg)(s-r g)(1+g)(-1+g))$$

ugh[7]: cost of new integrand:

1347 additions + 6886 multiplications + 345 divisions + 364 functions + 3 subscripts

ugh[7]: simplifying sin/cos part of new integrand...

ugh[8]: cost of simplified new integrand:

43 additions + 85 multiplications + 16 divisions + 13 functions + 3 subscripts

ugh[8]: new definite integral:

$$\int_{v_1}^{v_2} \left(\frac{1}{2} \frac{\cos(r(-1+g)x)}{(-1+g)^2 r^2} - \frac{1}{2} \frac{\cos(r(1+g)x)}{(1+g)^2 r^2} + \frac{1}{16} \frac{\cos(2rgx)}{g^2 r^2} + \frac{1}{8} \frac{\cos(2sx)}{s^2} \right. \\ \left. - \frac{1}{32} \frac{\cos((2s+2r)x)}{(s+r)^2} - \frac{1}{32} \frac{\cos((2s-2r)x)}{(s-r)^2} - \frac{1}{32} \frac{\cos((2s+2rg)x)}{(s+rg)^2} + \frac{1}{16} \frac{\cos(2rx)}{r^2} \right. \\ \left. - \frac{1}{4} \frac{\cos((2s-r+rg)x)}{(2s-r+rg)^2} - \frac{1}{4} \frac{\cos((2s+r-rg)x)}{(2s+r-rg)^2} + \frac{1}{4} \frac{\cos((2s+r+rg)x)}{(2s+r+rg)^2} \right. \\ \left. + \frac{1}{4} \frac{\cos((2s-r-rg)x)}{(2s-r-rg)^2} - \frac{1}{32} \frac{\cos((2s-2rg)x)}{(s-rg)^2} + \frac{1}{4} x^2 \right) (2a_2 + 6xa_3 + 12x^2a_4) dx$$

ugh(%)

ugh[8]: polynomial factor p:

$$2a_2 + 6xa_3 + 12x^2a_4$$

ugh[8]: location of p in integrand:

[2]

ugh[8]: integrating by parts...

ugh[14]: cost of evaluated part of integral:

9515 additions + 83564 multiplications + 2298 divisions + 2562 functions + 2686 subscripts

ugh[14]: simplifying...

ugh[38]: cost of evaluated part of integral:

146 additions + 385 multiplications + 32 divisions + 26 functions + 188 subscripts

ugh[38]: evaluated part of integral is stored in parts[2]

ugh[38]: new polynomial factor p:

$$6a_3 + 24xa_4$$

ugh[38]: location of p in integrand:

[2]

ugh[38]: full non-sin/cos term:

$$\frac{1}{32}(6a_3 + 24xa_4) / ((-1+g)^2 r^2 (1+g)^2 g^2 s^2 (s+r)^2 (s-r)^2 (s+rg)^2 (2s-r+rg)^2$$

$$(2s+r-rg)^2 (2s+r+rg)^2 (2s-r-rg)^2 (s-rg)^2$$

ugh[39]: cost of new integrand:

4756 additions + 41779 multiplications + 1149 divisions + 1281 functions + 2 subscripts

ugh[39]: simplifying sin/cos part of new integrand...

ugh[45]: cost of simplified new integrand:

42 additions + 99 multiplications + 16 divisions + 13 functions + 2 subscripts

ugh[45]: new definite integral:

$$\begin{aligned}
& - \int_{v_1}^{v_2} \left(\frac{1}{32} \frac{\sin(2rgx)}{g^3 r^3} + \frac{1}{16} \frac{\sin(2sx)}{s^3} - \frac{1}{64} \frac{\sin((2s+2r)x)}{(s+r)^3} - \frac{1}{64} \frac{\sin((2s-2r)x)}{(s-r)^3} \right. \\
& \quad - \frac{1}{64} \frac{\sin((2s+2rg)x)}{(s+rg)^3} + \frac{1}{32} \frac{\sin(2rx)}{r^3} - \frac{1}{4} \frac{\sin((2s-r+rg)x)}{(2s-r+rg)^3} - \frac{1}{4} \frac{\sin((2s+r-rg)x)}{(2s+r-rg)^3} \\
& \quad + \frac{1}{4} \frac{\sin((2s+r+rg)x)}{(2s+r+rg)^3} + \frac{1}{4} \frac{\sin((2s-r-rg)x)}{(2s-r-rg)^3} - \frac{1}{64} \frac{\sin((2s-2rg)x)}{(s-rg)^3} - \frac{1}{2} \frac{\sin(r(1+g)x)}{(1+g)^3 r^3} \\
& \quad \left. + \frac{1}{2} \frac{\sin(r(-1+g)x)}{r^3 (-1+g)^3} + \frac{1}{12} x^3 \right) (6a_3 + 24xa_4) dx
\end{aligned}$$

ugh(%)

ugh[45]: polynomial factor p:

$$6a_3 + 24xa_4$$

ugh[45]: location of p in integrand:

[2]

ugh[45]: integrating by parts...

ugh[62]: cost of evaluated part of integral:

20389 additions + 250998 multiplications + 4874 divisions + 5436 functions + 5662 subscripts

ugh[62]: simplifying...

ugh[147]: cost of evaluated part of integral:

117 additions + 333 multiplications + 32 divisions + 26 functions + 129 subscripts

ugh[147]: evaluated part of integral is stored in parts[3]

ugh[147]: no new polynomial factor!

ugh[150]: cost of new integrand:

10193 additions + 125496 multiplications + 2437 divisions + 2718 functions + subscripts

ugh[150]: simplifying sin/cos part of new integrand...

ugh[169]: cost of simplified new integrand:

41 additions + 126 multiplications + 16 divisions + 13 functions + 14 subscripts

ugh[169]: new definite integral:

$$\begin{aligned}
& \int_{v_1}^{v_2} \left(-12 \frac{a_4 \cos(r(-1+g)x)}{(-1+g)^4 r^4} + 12 \frac{a_4 \cos(r(1+g)x)}{(1+g)^4 r^4} - \frac{3}{8} \frac{a_4 \cos(2rgx)}{r^4 g^4} - \frac{3}{4} \frac{a_4 \cos(2sx)}{s^4} \right. \\
& \quad \left. + \frac{3}{16} \frac{a_4 \cos((2s+2r)x)}{(s+r)^4} + \frac{3}{16} \frac{a_4 \cos((2s-2r)x)}{(s-r)^4} + \frac{3}{16} \frac{a_4 \cos((2s+2rg)x)}{(s+rg)^4} - \frac{3}{8} \frac{a_4 \cos(2rx)}{r^4} \right)
\end{aligned}$$

$$+ 6 \frac{a_4 \cos((2s - r + rg)x)}{(2s - r + rg)^4} + 6 \frac{a_4 \cos((2s + r - rg)x)}{(2s + r - rg)^4} - 6 \frac{a_4 \cos((2s + r + rg)x)}{(2s + r + rg)^4}$$

$$- 6 \frac{a_4 \cos((2s - r - rg)x)}{(2s - r - rg)^4} + \frac{3}{16} \frac{a_4 \cos((2s - 2rg)x)}{(s - rg)^4} + \frac{1}{2} a_4 x^4 dx$$

last_integral := %

kernelopts(bytesalloc)

1024.0²

35.61847687

We see that we needed over 35 MB of memory to perform the simplifications. This last integral leads to a fatal explosion of terms if evaluated as a whole:

value(last_integral)
System error, ran out of memory
 Warning, computation interrupted

This is on a machine with more than 110 MB of memory available for use. Thwarted again by the easy approach, we will instead do this last integral in pieces.

pieces := [op(op(1, last_integral))]

nops(pieces)

14

time0 := time()

k := 4

for *p* **in** *pieces* **do** *parts*_{*k*} := factor $\left(\text{combine} \left(\int_{v_1}^{v_2} p \, dx, \text{trig} \right) \right)$; *k* := *k* + 1 **od**

Much better. Now combine those pieces.

sump := 0;

for *k* **from** 4 **to** 3 + *nops(pieces)* **do** *sump* := *sump* + *parts*_{*k*} **od**;

*parts*₄ := collect(*sump*, [seq(*a*_{*k*}, *k* = 0 .. 4), sin, cos, s], factor)

There, we've done the *entire* sensitivity integral. Let's see what we've got in terms of bloat.

*cost(parts*₀)

91 additions + 164 multiplications + 28 divisions + 26 functions + 46 subscripts

*cost(parts*₁)

174 additions + 466 multiplications + 32 divisions + 26 functions + 244 subscripts

```

cost(parts2)
    146 additions + 385 multiplications + 32 divisions + 26 functions + 188 subscripts

cost(parts3)
    117 additions + 333 multiplications + 32 divisions + 26 functions + 129 subscripts

cost(parts4)
    87 additions + 258 multiplications + 32 divisions + 26 functions + 37 subscripts

```

This doesn't look all that bad. There's one round more of tweaking to do.

We can pretty easily further simplify the last integral, *parts[4]*, by collecting on the factors that are common to the "v₁" sines and cosines and the "v₂" sines and cosines. Here is a sequence of Maple commands to accomplish this.

```

time0 := time():
verbosity := 1:
debug_print(``, `Here we go...`, 0);

# use a temporary for safety
tmp4 := parts[4];

# select the denominators we're interested in
Set_trig := select( has, indets(tmp4), {sin,cos} ):
Set_A := select( has, indets(tmp4), {nu[1]} ):
Set_A_trig := Set_trig intersect Set_A:
Set_denoms := {}:
for p in Set_A_trig do
    Set_denoms := Set_denoms union {select(has,coeff(tmp4,p,1),{s,g})};
od:
Set_denoms := Set_denoms minus {0,1}:
debug_print(``, `Set of denominators for parts[4]:`, 0, Set_denoms);

# substitute temporaries for the denominators
debug_print(``, `simplifying...`, 0);
Tlist := []:
denoms := convert(Set_denoms, list):
for k from 1 to nops(denoms) do
    tmp4 := subs( denoms[k]=T.k, tmp4 );
    Tlist := [op(Tlist), T.k]:
od:
debug_print(``, `expression with placeholders:`, 2, tmp4);

# now collect on the denominators
tmp4 := collect( tmp4, [a[4], op(Tlist)], factor );

# replace the denominator placeholders
for k from 1 to nops(denoms) do
    tmp4 := subs( T.k=denoms[k], tmp4 );
od:
debug_print(``, `simplified expression:`, 1, tmp4);

debug_print(``, `cost before simplification:`, 0, cost(parts[4]));
debug_print(``, `cost after simplification:`, 0, cost(tmp4));

```

```
# make sure we've not screwed up anything
debug_print(``, `error check: `, 0, factor(expand(tmp4-parts[4])));
```

[0] Here we go...

[0] Set of denominators for parts[4]:

$$\left\{ \frac{1}{(2s+r(1+g))^5}, \frac{1}{(s-r g)^5}, \frac{1}{(1+g)^5}, \frac{1}{(-1+g)^5}, \frac{1}{(s+r g)^5}, \frac{1}{(s-r)^5}, \frac{1}{g^5}, \frac{1}{s^5}, \frac{1}{(s+r)^5}, \right. \\ \left. \frac{1}{(2s-r(-1+g))^5}, \frac{1}{(2s+r(-1+g))^5}, \frac{1}{(2s-r(1+g))^5} \right\}$$

[0] simplifying...

[0] simplified expression:

$$\begin{aligned} & \left(\frac{-6 \sin((2s+r+r g)v_2) + 6 \sin((2s+r+r g)v_1)}{(2s+r(1+g))^5} \right. \\ & + \frac{-\frac{3}{32} \sin(2(s-r g)v_1) + \frac{3}{32} \sin(2(s-r g)v_2)}{(s-r g)^5} - 12 \frac{-\sin(r(1+g)v_2) + \sin(r(1+g)v_1)}{(1+g)^5 r^5} \\ & - 12 \frac{\sin(r(-1+g)v_2) - \sin(r(-1+g)v_1)}{(-1+g)^5 r^5} + \frac{-\frac{3}{32} \sin(2(s+r g)v_1) + \frac{3}{32} \sin(2(s+r g)v_2)}{(s+r g)^5} \\ & + \frac{-\frac{3}{32} \sin(2(s-r)v_1) + \frac{3}{32} \sin(2(s-r)v_2)}{(s-r)^5} - \frac{3}{16} \frac{\sin(2r g v_2) - \sin(2r g v_1)}{r^5 g^5} \\ & + \frac{-\frac{3}{8} \sin(2s v_2) + \frac{3}{8} \sin(2s v_1)}{s^5} + \frac{\frac{3}{32} \sin(2(s+r)v_2) - \frac{3}{32} \sin(2(s+r)v_1)}{(s+r)^5} \\ & + \frac{-6 \sin((2s+r-r g)v_1) + 6 \sin((2s+r-r g)v_2)}{(2s-r(-1+g))^5} \\ & + \frac{6 \sin((2s-r+r g)v_2) - 6 \sin((2s-r+r g)v_1)}{(2s+r(-1+g))^5} \\ & + \frac{-6 \sin((2s-r-r g)v_2) + 6 \sin((2s-r-r g)v_1)}{(2s-r(1+g))^5} \\ & \left. + \frac{1}{80} \frac{15 \sin(2r v_1) - 15 \sin(2r v_2) - 8r^5 v_1^5 + 8r^5 v_2^5}{r^5} \right) a_4 \end{aligned}$$

```

[0] cost before simplification:
    87 additions + 258 multiplications + 32 divisions + 26 functions + 37 subscripts
[0] cost after simplification:
    69 additions + 185 multiplications + 16 divisions + 26 functions + 29 subscripts
[0] error check:

```

0

Unfortunately, this simplification does not lead to more efficient results for *parts*[0..3].

Finally, combine the five parts of the integral.

```

defint := collect $\left(\sum_{k=0}^3 \text{parts}'_k\right) + \text{tmp4}, [\sin, \cos, v_1, v_2], \text{factor}$ 
cost(defint)

```

1155 additions + 3970 multiplications + 320 divisions + 64 functions + 1080 subscripts

Surely we can do better than this. Employ the cleanup routine that we wrote near the end of [section 2](#).

```

cleanup(defint, v_1, v_2)
cleanup[0]: number of unique sin/cos terms:

```

64

```

cleanup[0]: arg list:
[2 s - r + r g, 2 s - r - r g, 2 s + r + r g, 2 s + r - r g, s - r g, 2 s - 2 r g, 2 s + 2 r g, -r + r g, r + r g,
s + r g, r (-1 + g), r (1 + g), 2 r g, 2 s - 2 r, s - r, 2 s + 2 r, s + r, r g, 2 s, 2 r, r, s]
cleanup[0]: simplifying...
cleanup[9]: error check:

```

0

cost(%)

885 additions + 2287 multiplications + 160 divisions + 64 functions + 1039 subscripts

defint := S(r, s, g, T, v₁, v₂) = %%

This is much better, though still somewhat bulky. Let's determine the cost of an optimized expression sequence that would constitute the body of a fortran subroutine if we were to construct one now.

```

cost(optimize(rhs(defint)))

```

409 additions + 632 multiplications + 87 divisions + 64 functions + 148 subscripts + 346 assignments

Compare this with the equivalent for the indefinite integration result (remember to multiply by two).

2 cost(optimize(rhs(*indef*)))

238 additions + 372 multiplications + 182 divisions + 52 functions + 80 subscripts + 216 assignments

The definite integral result is not as efficient as two calls to the indefinite integral subroutine from [section 2](#), though we did surprisingly well, considering the complexity of the integral and the mess generated by successive by-parts integrations. However, we can use the definite integration result as a check on the indefinite integration result, which we will do in the next section.

For reference, the definite integration result is shown in the [Appendix](#).

[-] 5. Comparison between Definite and Indefinite Integrations

Here is the grand test: are the definite and indefinite integrations equivalent? First, evaluate the indefinite integral from v_1 to v_2 :

```
subs(x = v2, rhs(defint)) - subs(x = v1, rhs(defint))
```

```
cost(%)
```

595 additions + 1528 multiplications + 160 divisions + 52 functions + 722 subscripts

Subtract the definite integration from this and simplify the horrible result.

```
expand(% - rhs(defint))
```

```
cost(%)
```

8687 additions + 48796 multiplications + 3120 divisions + 8816 functions + 13360 subscripts

```
collect(% - rhs(defint), [sin, cos, seq(ak, k = 0 .. 4), v1, v2], factor)
```

0

Whew!! That was worth the wait.

[-] 6. Singularity Surfaces in Geometry Factor Parameter Space

[-] 6.1. Series Approximations of the Integral

Notice from the explicit expressions in the [Appendix](#) that the solutions are invalid (they blow up) for small r and small s . In fact, in the geometry factor parameter space (s, r, g) there are several surfaces upon which the numerical evaluation of the integral expression is singular. Upon doing series expansions in the vicinity of the singular surfaces, however, the integral becomes trivial, as we shall see in what follows.

First we find the singular surfaces. Write a procedure that extracts all of the denominators in an expression. Then set those expressions to zero, determining the singular surfaces.

```
get_denoms := proc(expr)
local denoms, p;
```

```

denoms := { };
for p in expr do
    if hastype(p, `^`) then
        if type(p, `^`) then if op(2, p) < 0 then denoms := { op(1, p) } union denoms fi
        else denoms := denoms union procname(p)
        fi
    fi
    od;
    denoms
end

denoms := get_denoms(indef)

denoms := { 2 s + r - r g, s, 2 s + r + r g, r, 2 s - r - r g, s + r, 2 s - r + r g, -1 + g, g, s + r g, s - r,
           s - r g, 1 + g }

blowups := { };
for p in denoms do blowups := blowups union solve({p}) od;
blowups := blowups minus { s = s, r = r, g = g }

blowups := map(factor, blowups)

blowups := { s =  $\frac{1}{2}r(-1+g)$ , g = 1, g = 0, r = 0, s = 0, r = s, s =  $\frac{1}{2}r(1+g)$ , s =  $-\frac{1}{2}r(-1+g)$ ,
             g = -1, s =  $-\frac{1}{2}r(1+g)$ , s = r g, s = -r g, r = -s }

Expansions of the integral kernel integrand0


$$\sin(s x)^2 (\sin(r x) - \sin(r g x))^2$$


across the singular surfaces are

for p in blowups do
    q := series(integrand0, p, 2);
    if convert(%, polynom) = 0 then q := series(integrand0, p, 3) fi;
    print(q)
od


$$\sin\left(\frac{1}{2}r(-1+g)x\right)^2 (\sin(r x) - \sin(r g x))^2 +$$


$$2 \sin\left(\frac{1}{2}r(-1+g)x\right) \cos\left(\frac{1}{2}r(-1+g)x\right) x (\sin(r x) - \sin(r g x))^2 \left(s - \frac{1}{2}r(-1+g)\right) +$$


$$O\left(\left(s - \frac{1}{2}r(-1+g)\right)^2\right)$$


$$\sin(s x)^2 \cos(r x)^2 r^2 x^2 (-1+g)^2 + O((-1+g)^3)$$


```

$$\begin{aligned}
& \sin(sx)^2 \sin(rx)^2 - 2 \sin(sx)^2 \sin(rx) rx g + O(g^2) \\
& \sin(sx)^2 (x - gx)^2 r^2 + O(r^4) \\
& x^2 (\sin(rx) - \sin(gx))^2 s^2 + O(s^4) \\
& \sin(sx)^2 (\sin(sx) - \sin(gx))^2 + \\
& 2 \sin(sx)^2 (\sin(sx) - \sin(gx)) (\cos(sx)x - \cos(gx)gx) (r-s) + O((r-s)^2) \\
& \sin\left(\frac{1}{2}r(1+g)x\right)^2 (\sin(rx) - \sin(gx))^2 + \\
& 2 \sin\left(\frac{1}{2}r(1+g)x\right) \cos\left(\frac{1}{2}r(1+g)x\right) x (\sin(rx) - \sin(gx))^2 \left(s - \frac{1}{2}r(1+g)\right) + \\
& O\left(\left(s - \frac{1}{2}r(1+g)\right)^2\right) \\
& \sin\left(\frac{1}{2}r(-1+g)x\right)^2 (\sin(rx) - \sin(gx))^2 - \\
& 2 \sin\left(\frac{1}{2}r(-1+g)x\right) \cos\left(\frac{1}{2}r(-1+g)x\right) x (\sin(rx) - \sin(gx))^2 \left(s + \frac{1}{2}r(-1+g)\right) + \\
& O\left(\left(s + \frac{1}{2}r(-1+g)\right)^2\right) \\
& 4 \sin(sx)^2 \sin(rx)^2 - 4 \sin(sx)^2 \sin(rx) \cos(rx) rx (1+g) + O((1+g)^2) \\
& \sin\left(\frac{1}{2}r(1+g)x\right)^2 (\sin(rx) - \sin(gx))^2 - \\
& 2 \sin\left(\frac{1}{2}r(1+g)x\right) \cos\left(\frac{1}{2}r(1+g)x\right) x (\sin(rx) - \sin(gx))^2 \left(s + \frac{1}{2}r(1+g)\right) + \\
& O\left(\left(s + \frac{1}{2}r(1+g)\right)^2\right) \\
& \sin(gx)^2 (\sin(rx) - \sin(gx))^2 + 2 \sin(gx) \cos(gx) x (\sin(rx) - \sin(gx))^2 (s - rg) \\
& + O((s - rg)^2) \\
& \sin(gx)^2 (\sin(rx) - \sin(gx))^2 - 2 \sin(gx) \cos(gx) x (\sin(rx) - \sin(gx))^2 (s + rg) \\
& + O((s + rg)^2) \\
& \sin(sx)^2 (-\sin(sx) + \sin(gx))^2 + \\
& 2 \sin(sx)^2 (-\sin(sx) + \sin(gx)) (\cos(sx)x - \cos(gx)gx) (s+r) + O((s+r)^2)
\end{aligned}$$

This is not encouraging, since, for most of these, integrating in frequency is similar in

difficulty to integrating the original *integral*,

$$\int_{v_1}^{v_2} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) \sin(s x)^2 (\sin(r x) - \sin(r g x))^2 dx$$

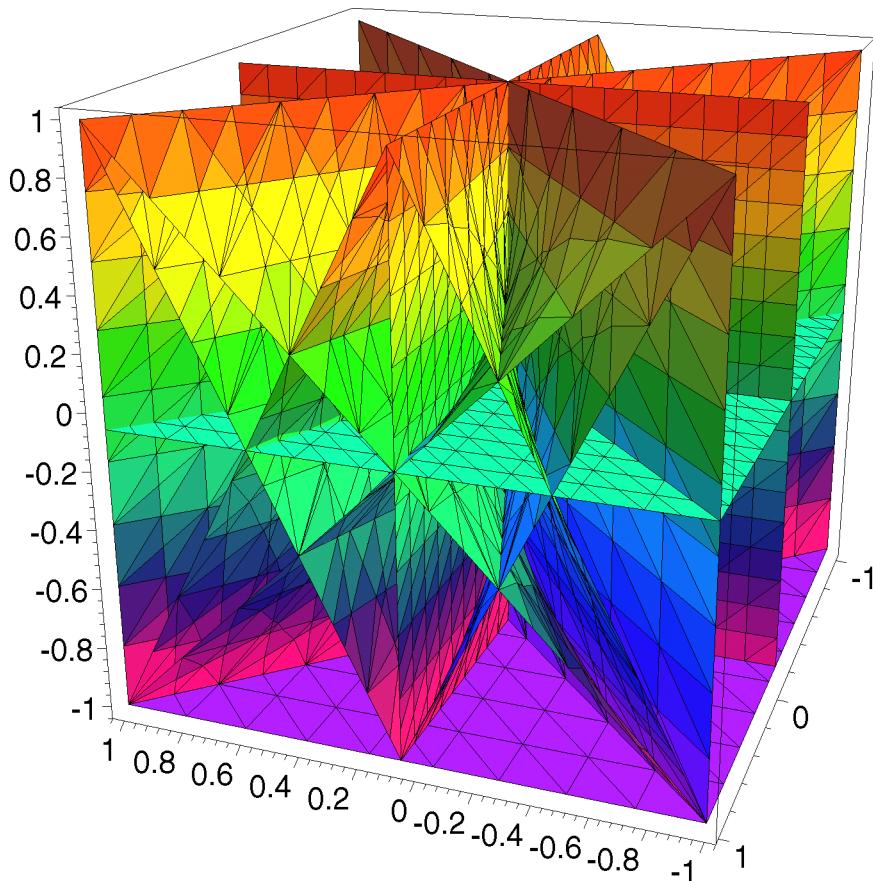
6.2. Graphical Representation of the Singular Surfaces

Let's look at a 3D plot of the singular surfaces. The plot below, though numerically incorrect in detail (s and r have been scaled), nonetheless correctly shows the topology of the surfaces. This is also not encouraging!

```
ssp := plots[implicitplot3d](denoms, s = -1 .. 1, r = -1 .. 1, g = -1 .. 1, scaling = constrained,  
style = patch, lightmodel = light2, orientation = [110, 70], title = "Singular Surfaces",  
labels = ["s", "r", "g"])
```

ssp

Singular Surfaces

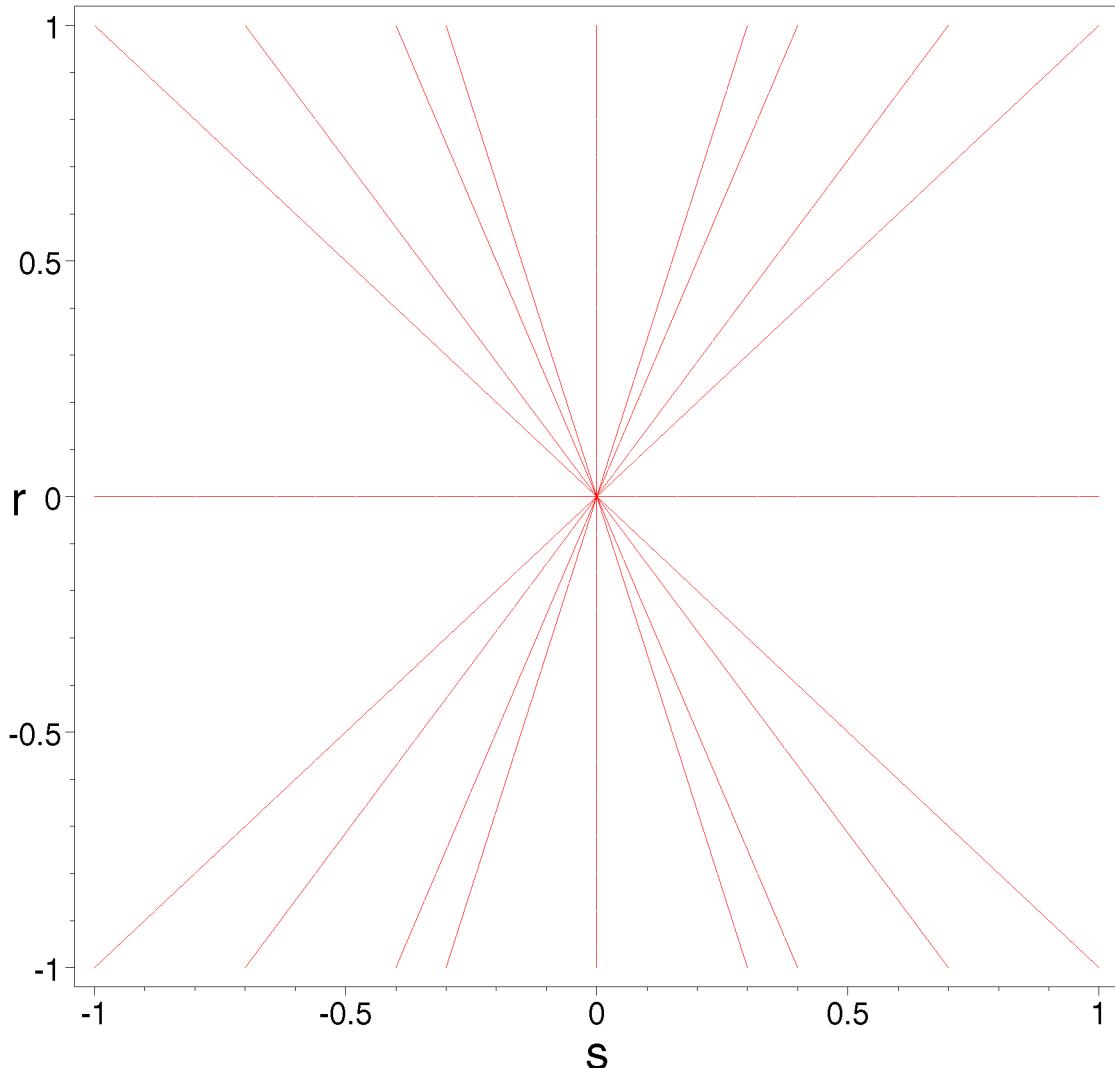


```

[ with(plotting)
[ gifplot(ssp, "SingularSurfaces", "d:/FAMEStuff/PhotonSensitivity/")
[ psplot(ssp, "SingularSurfaces", "d:/FAMEStuff/PhotonSensitivity/")
[ Let's look at the intersection of the surfaces with a plane of constant g.
plots implicitplot(subs(g = .4, denoms), s = -1 .. 1, r = -1 .. 1,
title = Singular Surface Intersections, g = 0.4)

```

Singular Surface Intersections, $g = 0.4$



6.3. Special Case: $r = s = 0$

6.3.1. Series Expansion

For the special case $r = s = 0$, then integral

$$S(r, s, g, T, v_1, v_2) = \int_{v_1}^{v_2} integrand_1 dx$$

$$S(r, s, g, T, v_1, v_2) =$$

$$\int_{v_1}^{v_2} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) \sin(s x)^2 (\sin(r x) - \sin(r g x))^2 dx$$

[becomes, to fourth order in combinations of s and r ,

$$S_4(r, s, g, T, v_1, v_2) = \int_{v_1}^{v_2} factor(expansion(intrand_1, [s, r], 4)) dx$$

$$S_4(r, s, g, T, v_1, v_2) = \int_{v_1}^{v_2} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) x^4 (-1 + g)^2 r^2 s^2 dx$$

[(Indeed, this fourth-order term is the first nonzero one to appear.) Hence we have

factor(value(%))

$$\begin{aligned} S_4(r, s, g, T, v_1, v_2) = & \frac{1}{2520} s^2 r^2 (-1 + g)^2 (-v_1 + v_2) \left(504 v_2^2 v_1^2 a_0^2 + 315 v_2^7 a_3^7 \right. \\ & + 360 v_2^6 a_2^2 + 420 v_2^5 a_1^5 + 504 v_2^4 a_0^4 + 280 a_4 v_2^8 + 360 a_2 v_1^6 + 315 a_3 v_1^7 \\ & + 280 a_4 v_1^8 + 504 v_1^4 a_0^4 + 420 a_1 v_1^5 + 315 v_2^6 v_1 a_3 + 280 v_2^7 v_1 a_4 \\ & + 280 v_2^5 v_1^3 a_4 + 315 v_2^5 v_1^2 a_3^2 + 360 v_2^5 v_1 a_2^2 + 280 v_2^6 v_1^2 a_4^2 + 315 v_2^2 a_3 v_1^5 \\ & + 360 v_2^2 a_2 v_1^4 + 420 v_2^2 a_1 v_1^3 + 280 v_2^3 a_4 v_1^5 + 315 v_2^3 a_3 v_1^4 + 360 v_2^3 a_2 v_1^3 \\ & + 420 v_2^3 a_1 v_1^2 + 504 v_2^3 v_1 a_0^2 + 280 v_2^4 a_4 v_1^4 + 315 v_2^4 a_3 v_1^3 + 360 v_2^4 a_2 v_1^2 \\ & + 420 v_2^4 a_1 v_1 + 280 v_2^4 a_4 v_1^7 + 315 v_2^4 a_3 v_1^6 + 360 v_2^4 a_2 v_1^5 + 420 v_2^4 a_1 v_1^4 \\ & \left. + 504 v_2^3 v_1^3 a_0^2 + 280 v_2^2 a_4 v_1^6 \right) \end{aligned}$$

Grab the messy term and clean it up.

`select(has, rhs(%), a0)`

$$\begin{aligned}
 & 504 v_2^2 v_1^2 a_0 + 315 v_2^7 a_3 + 360 v_2^6 a_2 + 420 v_2^5 a_1 + 504 v_2^4 a_0 + 280 a_4 v_2^8 \\
 & + 360 a_2 v_1^6 + 315 a_3 v_1^7 + 280 a_4 v_1^8 + 504 v_1^4 a_0 + 420 a_1 v_1^5 + 315 v_2^6 v_1 a_3 \\
 & + 280 v_2^7 v_1 a_4 + 280 v_2^5 v_1^3 a_4 + 315 v_2^5 v_1^2 a_3 + 360 v_2^5 v_1 a_2 + 280 v_2^6 v_1^2 a_4 \\
 & + 315 v_2^2 a_3 v_1^5 + 360 v_2^2 a_2 v_1^4 + 420 v_2^2 a_1 v_1^3 + 280 v_2^3 a_4 v_1^5 + 315 v_2^3 a_3 v_1^4 \\
 & + 360 v_2^3 a_2 v_1^3 + 420 v_2^3 a_1 v_1^2 + 504 v_2^3 v_1 a_0 + 280 v_2^4 a_4 v_1^4 + 315 v_2^4 a_3 v_1^3 \\
 & + 360 v_2^4 a_2 v_1^2 + 420 v_2^4 a_1 v_1 + 280 v_2 a_4 v_1^7 + 315 v_2 a_3 v_1^6 + 360 v_2 a_2 v_1^5 \\
 & + 420 v_2 a_1 v_1^4 + 504 v_2 v_1^3 a_0 + 280 v_2^2 a_4 v_1^6
 \end{aligned}$$

`applyop(collect, location(%%, %), %%, [seq(ai, i = 0 .. 4)], factor)`

$$\begin{aligned}
 S_4(r, s, g, T, v_1, v_2) = & \frac{1}{2520} s^2 r^2 (-1 + g)^2 (-v_1 + v_2) \left(\right. \\
 & \left(504 v_2^4 + 504 v_2^3 v_1 + 504 v_2^2 v_1^2 + 504 v_2 v_1^3 + 504 v_1^4 \right) a_0 \\
 & + 420 (v_1 + v_2) \left(v_2^2 + v_2 v_1 + v_1^2 \right) \left(v_1^2 - v_2 v_1 + v_2^2 \right) a_1 + \left(\right. \\
 & 360 v_1^3 v_2^3 + 360 v_1^4 v_2^2 + 360 v_1^2 v_2^4 + 360 v_1 v_2^5 + 360 v_1^6 + 360 v_2^6 + 360 v_1^5 v_2 \\
 & \left. \right) a_2 + 315 (v_1 + v_2) \left(v_2^2 + v_1^2 \right) \left(v_2^4 + v_1^4 \right) a_3 \\
 & \left. + 280 \left(v_2^2 + v_2 v_1 + v_1^2 \right) \left(v_1^6 + v_1^3 v_2^3 + v_2^6 \right) a_4 \right)
 \end{aligned}$$

This has a certain pleasing symmetry of form. However, we might more usefully rewrite

it in terms of the fractional bandpass, $\delta = \frac{v_2 - v_1}{v_1}$. Doing so, we find

`factor(simplify(%%, {v2 - v1 / v1 = δ}, [v2]))`

`select(has, rhs(%), a0)`

applyop(*collect*, location(%%, %), %%, v_1 , *factor*)

$$S_4(r, s, g, T, v_1, v_1(1+\delta)) = \frac{1}{2520} s^2 r^2 v_1^5 \delta (-1+g)^2 \left(\begin{aligned} & 280 a_4 (\delta^2 + 3\delta + 3) (\delta^6 + 6\delta^5 + 15\delta^4 + 21\delta^3 + 18\delta^2 + 9\delta + 3) v_1^4 \\ & + 315 a_3 (\delta + 2) (\delta^2 + 2\delta + 2) (\delta^4 + 4\delta^3 + 6\delta^2 + 4\delta + 2) v_1^3 \\ & + 360 a_2 (\delta^6 + 7\delta^5 + 21\delta^4 + 35\delta^3 + 35\delta^2 + 21\delta + 7) v_1^2 \\ & + 420 a_1 (\delta + 2) (\delta^2 + 3\delta + 3) (\delta^2 + \delta + 1) v_1 + 504 a_0 (5 + 10\delta + 10\delta^2 + 5\delta^3 + \delta^4) \end{aligned} \right)$$

defint_x4 := %

6.3.2. Error Evaluation, or When Do I Switch?

Notice that, within the large multiplicative term, the combinations of coefficients a_k times the start frequency are well-behaved in the numerical sense. That is, because numerically we have $a_k v_1^k \sim 1$, we would prefer them to occur in like powers, and indeed that term is

of the form $\sum A_k(\delta) a_k v_1^k$. The above expression, then, gives the value of the integral near the coordinate axes $\{s=0, r=0\}$, accurate to fourth order. The remaining question is deciding where one should switch over from the full integral expression ([section 7](#)) to the near-axis approximation. To that end, we can evaluate the next term in the expansion and use that for error control.

We are interested in the ratio

$$\varepsilon = \frac{\int_{v_1}^{v_2} O(\zeta^6) dx - \int_{v_1}^{v_2} O(\zeta^4) dx}{\int_{v_1}^{v_2} O(\zeta^4) dx}$$

where $\zeta = (s, r)$. When this approaches some factor of one (say $\varepsilon \sim 0.01$) then we are in trouble with the lowest order expansion and should use the full integral. We may rewrite this as

$$\int_{v_1}^{v_2} \text{factor}(\text{expansion}(integrand_1, [s, r], 6) - \text{expansion}(integrand_1, [s, r], 4)) dx$$

$$\text{select}(has, \text{op}(1, \%), g^2 r^2)$$

$$\epsilon = \frac{\text{applyop}(collect, \text{location}(\%, \%), \%, s, \text{factor})}{\int_{v_1}^{v_2} \text{factor}(\text{expansion}(integrand_1, [s, r], 4)) dx}$$

$$\epsilon =$$

$$\int_{v_1}^{v_2} -\frac{1}{3} s^2 x^6 r^2 (-1 + g)^2 (s^2 + r^2 (g^2 + g + 1)) (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) dx$$

$$\int_{v_1}^{v_2} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) x^4 (-1 + g)^2 r^2 s^2 dx$$

Remove the geometry terms to the outside of the integrals, for convenience collecting them into the quantity $Q = s^2 + r^2 (g^2 + g + 1)$.

$$\text{simplify}(\%, \{s^2 + r^2 (g^2 + g + 1) = Q\})$$

$$\text{factor}(\text{combine}(\%, Int))$$

$$\epsilon = \frac{\int_{v_1}^{v_2} -Q x^6 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) dx}{\int_{v_1}^{v_2} 3 x^4 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) dx}$$

Evaluate the integrals and substitute $\delta = \frac{v_2 - v_1}{v_1}$ as before. We find the result

```

    simplify

```

$$F(v_1, v_2) = v_1^2 \frac{\sum_k A_k(\delta) v_1^k}{\sum_k B_k(\delta) v_1^k}$$

is purely a function of the frequencies (and coefficients a_i) and is well-behaved. Hence, the frequency function $F(v_1, v_2)$ need be calculated *only once per integration over the sky*, making evaluation of ϵ very inexpensive.

6.3.3. Fortran Subroutines

Now we create optimized fortran subroutines to calculate the fourth-order expansion, *defint_x4*

$$\begin{aligned} S_4(r, s, g, T, v_1, v_1(1 + \delta)) = & \frac{1}{2520} s^2 r^2 v_1^5 \delta (-1 + g)^2 \left(\right. \\ & 280 a_4 (\delta^2 + 3 \delta + 3) (\delta^6 + 6 \delta^5 + 15 \delta^4 + 21 \delta^3 + 18 \delta^2 + 9 \delta + 3) v_1^4 \\ & + 315 a_3 (\delta + 2) (\delta^2 + 2 \delta + 2) (\delta^4 + 4 \delta^3 + 6 \delta^2 + 4 \delta + 2) v_1^3 \\ & + 360 a_2 (\delta^6 + 7 \delta^5 + 21 \delta^4 + 35 \delta^3 + 35 \delta^2 + 21 \delta + 7) v_1^2 \\ & \left. + 420 a_1 (\delta + 2) (\delta^2 + 3 \delta + 3) (\delta^2 + \delta + 1) v_1 + 504 a_0 (5 + 10 \delta + 10 \delta^2 + 5 \delta^3 + \delta^4) \right) \end{aligned}$$

and the corresponding error function, ϵ . First we process the series approximation. Create a Maple procedure to evaluate the series approximation expression.

```
tmp := subs(seq(a_k = a.k, k = 0 .. 4), v_1 = v1, rhs(defint_x4))
```

F :=

```
optimize/makeproc([optimize(tmp, tryhard)], parameters = [delta, v1, seq(a.i, i = 0 .. 4), s, r, g])
```

The cost of the optimized expression sequence is

```
cost(optimize(tmp, tryhard))
```

32 additions + 52 multiplications + 10 assignments

Create a fortran subroutine from the Maple procedure.

```
fortran(F, optimized, mode = double, precision = double,
filename = "d:/FAMEStuff/PhotonSensitivity/expansion4.f")
```

Next, we split the error function into geometrical and frequency parts, so that the frequency part can be evaluated independently. Create a Maple procedure to evaluate the frequency function:

```
remove(has, rhs(error_expr), s)
```

$$\begin{aligned}
& 1 / \left(9240 a_4 (\delta^2 + 3\delta + 3) (\delta^6 + 6\delta^5 + 15\delta^4 + 21\delta^3 + 18\delta^2 + 9\delta + 3) v_1^8 \right. \\
& + 10395 a_3 (\delta + 2) (\delta^2 + 2\delta + 2) (\delta^4 + 4\delta^3 + 6\delta^2 + 4\delta + 2) v_1^7 \\
& + 11880 a_2 (\delta^6 + 7\delta^5 + 21\delta^4 + 35\delta^3 + 35\delta^2 + 21\delta + 7) v_1^6 \\
& + 13860 a_1 (\delta + 2) (\delta^2 + 3\delta + 3) (\delta^2 + \delta + 1) v_1^5 \\
& \left. + 16632 a_0 (5 + 10\delta + 10\delta^2 + 5\delta^3 + \delta^4) v_1^4 \right)
\end{aligned}$$

tmp := subs(seq(a_k = a.k, k = 0 .. 4), v₁ = v1, %)

cost(optimize(tmp, tryhard))

31 additions + 47 multiplications + divisions + 10 assignments

(Remember this has to be evaluated only once per integration over the sky.)

G := optimize/makeproc([optimize(tmp, tryhard)], parameters = [δ, v1, seq(a.i, i = 0 .. 4)])

Now create the geometry function Maple procedure:

select(has, rhs(error_expr), s)

$$-2520 (s^2 + r^2 (g^2 + g + 1)) a_4$$

$$(\delta^{10} + 462\delta^5 + 55\delta^8 + 330\delta^3 + 330\delta^6 + 55\delta + 165\delta^7 + 11\delta^9 + 11 + 165\delta^2 + 462\delta^4)$$

$$v_1^{10} - 2772 (s^2 + r^2 (g^2 + g + 1)) a_3 (\delta + 2) (\delta^4 + 3\delta^3 + 4\delta^2 + 2\delta + 1)$$

$$(5 + 10\delta + 10\delta^2 + 5\delta^3 + \delta^4) v_1^9 - 3080$$

$$(s^2 + r^2 (g^2 + g + 1)) a_2 (\delta^2 + 3\delta + 3) (\delta^6 + 6\delta^5 + 15\delta^4 + 21\delta^3 + 18\delta^2 + 9\delta + 3) v_1^8$$

$$- 3465 (s^2 + r^2 (g^2 + g + 1)) a_1 (\delta + 2) (\delta^2 + 2\delta + 2) (\delta^4 + 4\delta^3 + 6\delta^2 + 4\delta + 2) v_1^7$$

$$- 3960 (s^2 + r^2 (g^2 + g + 1)) a_0 (\delta^6 + 7\delta^5 + 21\delta^4 + 35\delta^3 + 35\delta^2 + 21\delta + 7) v_1^6$$

tmp := subs(seq(a_k = a.k, k = 0 .. 4), v₁ = v1, %)

H :=

optimize/makeproc([optimize(% , tryhard)], parameters = [δ, v1, seq(a.i, i = 0 .. 4), s, r, g])

Finally, convert the Maple functions into fortran subroutines.

*fortran(G, optimized, mode = double, precision = double,
filename = "d:/FAMEStuff/PhotonSensitivity/error_freq.f")*

```
fortran(H, optimized, mode = double, precision = double,  
filename = "d:/FAMEStuff/PhotonSensitivity/error_geom.f")
```

For reference, here are the fortran subroutines.

```
fortran(F, optimized, mode = double, precision = double)  
    doubleprecision function F(delta,nul,a0,a1,a2,a3,a4,s,r,g)  
    doubleprecision delta  
    doubleprecision nul  
    doubleprecision a0  
    doubleprecision a1  
    doubleprecision a2  
    doubleprecision a3  
    doubleprecision a4  
    doubleprecision s  
    doubleprecision r  
    doubleprecision g  
  
    doubleprecision t180  
    doubleprecision t181  
    doubleprecision t182  
    doubleprecision t185  
    doubleprecision t186  
    doubleprecision t188  
    doubleprecision t190  
    doubleprecision t194  
    doubleprecision t3  
    doubleprecision t4  
    doubleprecision t8  
  
    t194 = nul*(delta+2.D0)  
    t185 = delta**2  
    t190 = t185*delta  
    t186 = nul**2  
    t188 = t186**2  
    t182 = t185**2  
    t181 = t190**2  
    t180 = t185*t190  
    t3 = s**2  
    t4 = r**2  
    t8 = (-1.D0+g)**2.D0  
    F = t3*t4*nul*t188*delta*t8*(504.D0*a0*(5.D0+10.D0*delta+10.D0*t  
#185+5.D0*t190+t182)+(420.D0*a1*(t185+delta+1.D0)*t194+280.D0*a4*(t  
#181+6.D0*t180+15.D0*t182+21.D0*t190+18.D0*t185+9.D0*delta+3.D0)*t1  
#88)*(t185+3.D0*delta+3.D0)+(315.D0*a3*(t185+2.D0*delta+2.D0)*(t182  
#+4.D0*t190+6.D0*t185+4.D0*delta+2.D0)*t194+360.D0*a2*(t181+7.D0*t1  
#80+21.D0*t182+35.D0*t190+35.D0*t185+21.D0*delta+7.D0))*t186)/2520.  
#D0  
    return  
end  
  
fortran(G, optimized, mode = double, precision = double)  
    doubleprecision function G(delta,nul,a0,a1,a2,a3,a4)  
    doubleprecision delta  
    doubleprecision nul  
    doubleprecision a0  
    doubleprecision a1  
    doubleprecision a2  
    doubleprecision a3
```

```

doubleprecision a4

doubleprecision t229
doubleprecision t231
doubleprecision t232
doubleprecision t235
doubleprecision t236
doubleprecision t237
doubleprecision t238
doubleprecision t239
doubleprecision t242
doubleprecision t243

t235 = delta**2
t243 = t235**2
t242 = delta*t235
t236 = nul**2
t237 = t236**2
t238 = nul*t237
t239 = nul*t238
t232 = t242**2
t231 = delta*t243
t229 = t235+3.D0*delta+3.D0
G = 1.D0/(11880.D0*a2*(t232+7.D0*t231+21.D0*t243+35.D0*t242+35.D
#0*t235+21.D0*delta+7.D0)*t239+(10395.D0*a3*(t235+2.D0*delta+2.D0)*
#(t243+4.D0*t242+6.D0*t235+4.D0*delta+2.D0)*nul*t239+13860.D0*a1*t2
#29*(t235+delta+1.D0)*t238)*(delta+2.D0)+(16632.D0*a0*(5.D0+10.D0*d
#elta+10.D0*t235+5.D0*t242+t243)+9240.D0*a4*t229*(t232+6.D0*t231+15
#.D0*t243+21.D0*t242+18.D0*t235+9.D0*delta+3.D0)*t237)*t237)
return
end

fortran(H, optimized, mode = double, precision = double)
doubleprecision function H(delta,nul,a0,a1,a2,a3,a4,s,r,g)
doubleprecision delta
doubleprecision nul
doubleprecision a0
doubleprecision a1
doubleprecision a2
doubleprecision a3
doubleprecision a4
doubleprecision s
doubleprecision r
doubleprecision g

doubleprecision t369
doubleprecision t373
doubleprecision t374
doubleprecision t376
doubleprecision t377
doubleprecision t378
doubleprecision t379
doubleprecision t381
doubleprecision t382
doubleprecision t383
doubleprecision t384
doubleprecision t386
doubleprecision t63
doubleprecision t64

```

```

doubleprecision t65

t373 = delta**2
t382 = t373**2
t386 = t382**2
t381 = delta*t373
t384 = t381**2
t383 = delta*t382
t374 = nul**2
t376 = nul**2*t374**2
t377 = nul*t376
t378 = nul*t377
t379 = nul*t378
t369 = 2.D0*delta
t63 = s**2
t64 = r**2
t65 = g**2
H = (-2520.D0*a4*(330.D0*t384+165.D0*t373+55.D0*t386+330.D0*t381
#+462.D0*t382+11.D0+(462.D0+t383)*t383+(11.D0*t386+55.D0+165.D0*t38
#4)*delta)*nul*t379-3080.D0*a2*(t373+3.D0*delta+3.D0)*(t384+6.D0*t3
#83+15.D0*t382+21.D0*t381+18.D0*t373+9.D0*delta+3.D0)*t378-3960.D0*
#a0*(t384+7.D0*t383+21.D0*t382+35.D0*t381+35.D0*t373+21.D0*delta+7.
#D0)*t376+(-2772.D0*a3*(t382+3.D0*t381+4.D0*t373+t369+1.D0)*(5.D0+1
#.D0*delta+10.D0*t373+5.D0*t381+t382)*t379-3465.D0*a1*(t373+t369+2
#.D0)*(t382+4.D0*t381+6.D0*t373+4.D0*delta+2.D0)*t377)*(delta+2.D0)
#)*(t63+t64*(t65+g+1.D0))
    return
end

```

[-] Appendix: Integration Result Expressions

Here is the full expression for the *indefinite* integration result calculated in [section 2](#):

indef

$S(r, s, g, T) =$

$$\begin{aligned}
& \left(\frac{-\frac{1}{4}a_1x - \frac{1}{4}a_0 - \frac{1}{4}a_3x^3 - \frac{1}{4}a_2x^2 - \frac{1}{4}a_4x^4}{2s - r - rg} + \frac{\frac{3}{2}xa_3 + 3x^2a_4 + \frac{1}{2}a_2}{(2s - r - rg)^3} - 6 \frac{a_4}{(2s - r - rg)^5} \right) \\
& \sin((2s - r - rg)x) + \\
& \left(\frac{\frac{1}{4}a_1x + \frac{1}{4}a_0 + \frac{1}{4}a_3x^3 + \frac{1}{4}a_2x^2 + \frac{1}{4}a_4x^4}{2s - r + rg} + \frac{-\frac{3}{2}xa_3 - 3x^2a_4 - \frac{1}{2}a_2}{(2s - r + rg)^3} + 6 \frac{a_4}{(2s - r + rg)^5} \right) \\
& \sin((2s - r + rg)x) + \\
& \left(\frac{\frac{1}{4}a_1x + \frac{1}{4}a_0 + \frac{1}{4}a_3x^3 + \frac{1}{4}a_2x^2 + \frac{1}{4}a_4x^4}{2s + r - rg} + \frac{-\frac{3}{2}xa_3 - 3x^2a_4 - \frac{1}{2}a_2}{(2s + r - rg)^3} + 6 \frac{a_4}{(2s + r - rg)^5} \right)
\end{aligned}$$

$$\begin{aligned}
& \sin((2s + r - rg)x) + \left(\frac{-\frac{1}{2}xa_2 - \frac{3}{4}x^2a_3 - \frac{1}{4}a_1 - x^3a_4}{(2s + r + rg)^2} + \frac{6xa_4 + \frac{3}{2}a_3}{(2s + r + rg)^4} \right) \cos((2s + r + rg)x) \\
& + \left(\frac{-\frac{1}{2}xa_2 - \frac{3}{4}x^2a_3 - \frac{1}{4}a_1 - x^3a_4}{(2s - r - rg)^2} + \frac{6xa_4 + \frac{3}{2}a_3}{(2s - r - rg)^4} \right) \cos((2s - r - rg)x) \\
& + \left(\frac{\frac{1}{2}xa_2 + \frac{3}{4}x^2a_3 + \frac{1}{4}a_1 + x^3a_4}{(2s - r + rg)^2} + \frac{-6xa_4 - \frac{3}{2}a_3}{(2s - r + rg)^4} \right) \cos((2s - r + rg)x) + \\
& \left(\frac{\frac{1}{16}a_1x + \frac{1}{16}a_0 + \frac{1}{16}a_3x^3 + \frac{1}{16}a_2x^2 + \frac{1}{16}a_4x^4}{s - rg} + \frac{-\frac{3}{32}xa_3 - \frac{3}{16}x^2a_4 - \frac{1}{32}a_2}{(s - rg)^3} + \frac{3}{32} \frac{a_4}{(s - rg)^5} \right) \\
& \sin((2s - 2rg)x) \\
& + \left(\frac{\frac{1}{16}xa_2 + \frac{3}{32}x^2a_3 + \frac{1}{32}a_1 + \frac{1}{8}x^3a_4}{(s - rg)^2} + \frac{-\frac{3}{16}xa_4 - \frac{3}{64}a_3}{(s - rg)^4} \right) \cos((2s - 2rg)x) + \\
& \left(\frac{-\frac{1}{4}a_1x - \frac{1}{4}a_0 - \frac{1}{4}a_3x^3 - \frac{1}{4}a_2x^2 - \frac{1}{4}a_4x^4}{2s + r + rg} + \frac{\frac{3}{2}xa_3 + 3x^2a_4 + \frac{1}{2}a_2}{(2s + r + rg)^3} - 6 \frac{a_4}{(2s + r + rg)^5} \right) \\
& \sin((2s + r + rg)x) + \left(\frac{\frac{1}{2}xa_2 + \frac{3}{4}x^2a_3 + \frac{1}{4}a_1 + x^3a_4}{(2s + r - rg)^2} + \frac{-6xa_4 - \frac{3}{2}a_3}{(2s + r - rg)^4} \right) \cos((2s + r - rg)x) \\
& + \left(-\frac{1}{8} \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}{gr} + \frac{1}{16} \frac{3xa_3 + 6x^2a_4 + a_2}{g^3r^3} - \frac{3}{16} \frac{a_4}{g^5r^5} \right) \sin(2rgx) \\
& + \left(\frac{-\frac{1}{8}xa_2 - \frac{3}{16}x^2a_3 - \frac{1}{16}a_1 - \frac{1}{4}x^3a_4}{r^2} + \frac{\frac{3}{8}xa_4 + \frac{3}{32}a_3}{r^4} \right) \cos(2rx) \\
& + \left(-\frac{1}{16} \frac{2xa_2 + 3x^2a_3 + a_1 + 4x^3a_4}{g^2r^2} + \frac{3}{32} \frac{4xa_4 + a_3}{g^4r^4} \right) \cos(2rgx) \\
& + \left(\frac{1}{2} \frac{2xa_2 + 3x^2a_3 + a_1 + 4x^3a_4}{(1+g)^2r^2} - 3 \frac{4xa_4 + a_3}{(1+g)^4r^4} \right) \cos((r + rg)x)
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{2} \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{(1+g)r} - \frac{3x a_3 + 6x^2 a_4 + a_2}{(1+g)^3 r^3} + 12 \frac{a_4}{(1+g)^5 r^5} \right) \sin((r+r g)x) + \\
& \left(-\frac{1}{2} \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{(-1+g)r} + \frac{3x a_3 + 6x^2 a_4 + a_2}{(-1+g)^3 r^3} - 12 \frac{a_4}{(-1+g)^5 r^5} \right) \sin((-r+r g)x) \\
& + \left(\frac{\frac{1}{16}x a_2 + \frac{3}{32}x^2 a_3 + \frac{1}{32}a_1 + \frac{1}{8}x^3 a_4}{(s+r)^2} + \frac{-\frac{3}{16}x a_4 - \frac{3}{64}a_3}{(s+r)^4} \right) \cos((2s+2r)x) + \\
& \left(\frac{\frac{1}{16}a_1 x + \frac{1}{16}a_0 + \frac{1}{16}a_3 x^3 + \frac{1}{16}a_2 x^2 + \frac{1}{16}a_4 x^4}{s+r} + \frac{-\frac{3}{32}x a_3 - \frac{3}{16}x^2 a_4 - \frac{1}{32}a_2}{(s+r)^3} + \frac{3}{32} \frac{a_4}{(s+r)^5} \right) \\
& \sin((2s+2r)x) + \left(\frac{\frac{1}{16}x a_2 + \frac{3}{32}x^2 a_3 + \frac{1}{32}a_1 + \frac{1}{8}x^3 a_4}{(s-r)^2} + \frac{-\frac{3}{16}x a_4 - \frac{3}{64}a_3}{(s-r)^4} \right) \cos((2s-2r)x) + \\
& \left(\frac{\frac{1}{16}a_1 x + \frac{1}{16}a_0 + \frac{1}{16}a_3 x^3 + \frac{1}{16}a_2 x^2 + \frac{1}{16}a_4 x^4}{s-r} + \frac{-\frac{3}{32}x a_3 - \frac{3}{16}x^2 a_4 - \frac{1}{32}a_2}{(s-r)^3} + \frac{3}{32} \frac{a_4}{(s-r)^5} \right) \\
& \sin((2s-2r)x) + \left(\frac{\frac{1}{16}x a_2 + \frac{3}{32}x^2 a_3 + \frac{1}{32}a_1 + \frac{1}{8}x^3 a_4}{(s+r g)^2} + \frac{-\frac{3}{16}x a_4 - \frac{3}{64}a_3}{(s+r g)^4} \right) \cos((2s+2r g)x) \\
& + \\
& \left(\frac{\frac{1}{16}a_1 x + \frac{1}{16}a_0 + \frac{1}{16}a_3 x^3 + \frac{1}{16}a_2 x^2 + \frac{1}{16}a_4 x^4}{s+r g} + \frac{-\frac{3}{32}x a_3 - \frac{3}{16}x^2 a_4 - \frac{1}{32}a_2}{(s+r g)^3} + \frac{3}{32} \frac{a_4}{(s+r g)^5} \right) \\
& \sin((2s+2r g)x) \\
& + \left(\frac{-\frac{1}{4}a_1 x - \frac{1}{4}a_0 - \frac{1}{4}a_3 x^3 - \frac{1}{4}a_2 x^2 - \frac{1}{4}a_4 x^4}{s} + \frac{\frac{3}{8}x a_3 + \frac{3}{4}x^2 a_4 + \frac{1}{8}a_2}{s^3} - \frac{3}{8} \frac{a_4}{s^5} \right) \sin(2s x) \\
& + \left(-\frac{1}{2} \frac{2x a_2 + 3x^2 a_3 + a_1 + 4x^3 a_4}{(-1+g)^2 r^2} + 3 \frac{4x a_4 + a_3}{(-1+g)^4 r^4} \right) \cos((-r+r g)x) \\
& + \left(-\frac{1}{4} \frac{x a_2 - \frac{3}{8}x^2 a_3 - \frac{1}{8}a_1 - \frac{1}{2}x^3 a_4}{s^2} + \frac{\frac{3}{4}x a_4 + \frac{3}{16}a_3}{s^4} \right) \cos(2s x)
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{-\frac{1}{8}a_1x - \frac{1}{8}a_0 - \frac{1}{8}a_3x^3 - \frac{1}{8}a_2x^2 - \frac{1}{8}a_4x^4}{r} + \frac{\frac{3}{16}xa_3 + \frac{3}{8}x^2a_4 + \frac{1}{16}a_2}{r^3} - \frac{\frac{3}{16}\frac{a_4}{r^5}}{} \right) \sin(2rx) \\
& + \frac{1}{120}x(20a_2x^2 + 60a_0 + 30a_1x + 15a_3x^3 + 12a_4x^4)
\end{aligned}$$

Here is the fortran subroutine that evaluates the *indefinite* integration result calculated in [section 2](#). It evaluates the integral at frequency $v = x$. Hence, the integral from v_1 to v_2 is calculated by two calls to the subroutine and subtracting:

$$F(v_2, a0, a1, a2, a3, a4, s, r, g) - F(v_1, a0, a1, a2, a3, a4, s, r, g)$$

```

fortran(sensitivity_expr, optimized, mode = double, precision = double)
doubleprecision function sensitivity_expr(x,a0,a1,a2,a3,a4,s,r,g)
doubleprecision x
doubleprecision a0
doubleprecision a1
doubleprecision a2
doubleprecision a3
doubleprecision a4
doubleprecision s
doubleprecision r
doubleprecision g

doubleprecision t1
doubleprecision t10
doubleprecision t101
doubleprecision t104
doubleprecision t105
doubleprecision t109
doubleprecision t11
doubleprecision t110
doubleprecision t113
doubleprecision t114
doubleprecision t115
doubleprecision t116
doubleprecision t117
doubleprecision t118
doubleprecision t119
doubleprecision t12
doubleprecision t120
doubleprecision t121
doubleprecision t122
doubleprecision t123
doubleprecision t124
doubleprecision t125
doubleprecision t126
doubleprecision t127
doubleprecision t128
doubleprecision t129
doubleprecision t13
doubleprecision t130
doubleprecision t131

```

doubleprecision t132
doubleprecision t133
doubleprecision t134
doubleprecision t135
doubleprecision t136
doubleprecision t137
doubleprecision t138
doubleprecision t139
doubleprecision t14
doubleprecision t140
doubleprecision t141
doubleprecision t142
doubleprecision t143
doubleprecision t146
doubleprecision t147
doubleprecision t149
doubleprecision t15
doubleprecision t151
doubleprecision t153
doubleprecision t154
doubleprecision t155
doubleprecision t156
doubleprecision t16
doubleprecision t160
doubleprecision t162
doubleprecision t163
doubleprecision t164
doubleprecision t165
doubleprecision t166
doubleprecision t167
doubleprecision t168
doubleprecision t169
doubleprecision t17
doubleprecision t176
doubleprecision t177
doubleprecision t179
doubleprecision t18
doubleprecision t185
doubleprecision t19
doubleprecision t192
doubleprecision t195
doubleprecision t199
doubleprecision t2
doubleprecision t20
doubleprecision t206
doubleprecision t209
doubleprecision t21
doubleprecision t213
doubleprecision t218
doubleprecision t22
doubleprecision t23
doubleprecision t230
doubleprecision t24
doubleprecision t242
doubleprecision t244
doubleprecision t25
doubleprecision t250
doubleprecision t254
doubleprecision t259

```
doubleprecision t26
doubleprecision t27
doubleprecision t270
doubleprecision t279
doubleprecision t28
doubleprecision t289
doubleprecision t29
doubleprecision t296
doubleprecision t3
doubleprecision t30
doubleprecision t304
doubleprecision t31
doubleprecision t311
doubleprecision t317
doubleprecision t325
doubleprecision t333
doubleprecision t338
doubleprecision t34
doubleprecision t342
doubleprecision t347
doubleprecision t35
doubleprecision t357
doubleprecision t36
doubleprecision t39
doubleprecision t4
doubleprecision t40
doubleprecision t45
doubleprecision t46
doubleprecision t47
doubleprecision t48
doubleprecision t49
doubleprecision t50
doubleprecision t51
doubleprecision t52
doubleprecision t53
doubleprecision t54
doubleprecision t56
doubleprecision t57
doubleprecision t58
doubleprecision t59
doubleprecision t6
doubleprecision t65
doubleprecision t69
doubleprecision t7
doubleprecision t73
doubleprecision t76
doubleprecision t77
doubleprecision t8
doubleprecision t81
doubleprecision t85
doubleprecision t89
doubleprecision t9
doubleprecision t93
doubleprecision t95
doubleprecision t97
doubleprecision t99
```

```
t58 = x**2
t56 = x*t58
```

```
t129 = a3*t56
t8 = t58**2
t130 = a4*t8
t177 = a0+t129+t130
t49 = r*g
t52 = 2.D0*s
t123 = t49+t52
t22 = -r+t123
t81 = t22**2
t164 = 1.D0/t81
t85 = g**2
t162 = 1.D0/t85
t45 = 1.D0+g
t93 = t45**2
t156 = 1.D0/t93
t48 = s+r
t89 = t48**2
t153 = 1.D0/t89
t46 = s-r
t97 = t46**2
t149 = 1.D0/t97
t24 = r+t123
t77 = t24**2
t146 = 1.D0/t77
t57 = 1.D0/r
t169 = 2.D0*x
t168 = 6.D0*a4
t167 = -t168
t122 = -t49+t52
t25 = r+t122
t73 = t25**2
t12 = t73**2
t166 = 1.D0/t12
t165 = 1.D0/t73
t23 = -r+t122
t69 = t23**2
t160 = 1.D0/t69
t59 = s**2
t155 = 1.D0/t59
t26 = t69**2
t154 = 1.D0/t26
t27 = t59**2
t151 = 1.D0/t27
t65 = r**2
t54 = 1.D0/t65
t28 = t65**2
t53 = 1.D0/t28
t34 = s-t49
t105 = t34**2
t143 = 1.D0/t105
t29 = t105**2
t142 = 1.D0/t29
t47 = -1.D0+g
t101 = t47**2
t141 = 1.D0/t101
t30 = t101**2
t140 = 1.D0/t30
t36 = s+t49
t109 = t36**2
```

```

t139 = 1.D0/t109
t31 = t109**2
t138 = 1.D0/t31
t137 = x*a2
t136 = a1*x
t135 = x*a3
t134 = x*a4
t51 = t57*t53
t133 = a4*t51
t125 = t58*a3
t128 = t56*a4
t132 = (2.D0*t137+3.D0*t125+a1+4.D0*t128)*t54
t126 = a2*t58
t131 = t57*(t136+t126+t177)
t127 = t58*a4
t124 = (4.D0*t134+a3)*t53
t39 = r*t169
t50 = t57*t54
t121 = (3.D0*t135+6.D0*t127+a2)*t50
t120 = 3.D0/32.D0*a3
t119 = 3.D0/16.D0*a3
t118 = 3.D0/32.D0*a4
t117 = a2/16.D0
t116 = 3.D0/16.D0*a4
t115 = a1/16.D0
t114 = 3.D0/16.D0*t58
t113 = t51*t116
t40 = s*t169
t35 = g*t39
t21 = -6.D0*t134-3.D0/2.D0*a3
t20 = (r+t49)*x
t19 = (-r+t49)*x
t18 = t48*t169
t17 = t46*t169
t16 = t24*x
t15 = t23*x
t14 = t22*x
t13 = t25*x
t11 = t34*t169
t10 = t36*t169
t9 = -3.D0/2.D0*t135-3.D0*t127-a2/2.D0
t7 = -x*t116-3.D0/64.D0*a3
t6 = t137/2.D0+3.D0/4.D0*t125+a1/4.D0+t128
t76 = -t136-a0-t129-t126-t130
t4 = t76/4.D0
t3 = -x*t120-a4*t114-a2/32.D0
t2 = x*t117+t58*t120+a1/32.D0+t128/8.D0
t1 = x*t115+t58*t117+t177/16.D0
t95 = dcos(t13)
t99 = t89**2
t104 = dcos(t18)
t110 = t85**2
t147 = dcos(t35)
t163 = dsin(t16)
t176 = dsin(t15)
t179 = t131/2.D0
t185 = dsin(t20)
t192 = dcos(t11)
t195 = t97**2

```

```

t199 = dcos(t17)
t206 = dsin(t13)
t209 = t81**2
t213 = dcos(t14)
t218 = dcos(t10)
t230 = dsin(t40)
t242 = dcos(t40)
t244 = (t6*t165+t21*t166)*t95+(t2*t153+t7/t99)*t104+(-t162*t132/
#16.D0+3.D0/32.D0/t110*t124)*t147+(t4+(-t9+t146*t167)*t146)/t24*t16
#3+(t4-t9*t160+t154*t167)/t23*t176+(t179+(-t121+12.D0*t156*t133)*t1
#56)*t185/t45+(t2*t143+t7*t142)*t192+(t2*t149+t7/t195)*t199+(-t4+t9
#*t165+t166*t168)/t25*t206+(t6*t164+t21/t209)*t213+(t2*t139+t7*t138
#)*t218+(t4+(3.D0/8.D0*t135+3.D0/4.D0*t127+a2/8.D0)*t155-3.D0/8.D0*
#a4*t151)/s*t230+((-t137/4.D0-3.D0/8.D0*t125-a1/8.D0-t128/2.D0)*t15
#5+(3.D0/4.D0*t134+t119)*t151)*t242
t250 = dcos(t19)
t254 = t93**2
t259 = dcos(t20)
t270 = dcos(t39)
t279 = dsin(t39)
t289 = dsin(t35)
t296 = dsin(t10)
t304 = dsin(t19)
t311 = dsin(t11)
t317 = dsin(t14)
t325 = dsin(t18)
t333 = dsin(t17)
t338 = t77**2
t342 = dcos(t16)
t347 = dcos(t15)
t357 = (-t141*t132/2.D0+3.D0*t140*t124)*t250+(t156*t132/2.D0-3.D
#0/t254*t124)*t259+((-t137/8.D0-a3*t114-t115-t128/4.D0)*t54+(3.D0/8
#.D0*t134+t120)*t53)*t270+(t76*t57/8.D0-t113+(3.D0/8.D0*t127+t117+x
##*t119)*t50)*t279+(-t131/8.D0+(t121/16.D0-t162*t113)*t162)/g*t289+(#
t1+t3*t139+t138*t118)/t36*t296+(-t179+t141*t121-12.D0*t140*t133)/t
#47*t304+(t1+t3*t143+t142*t118)/t34*t311+(-t4+(t9+t164*t168)*t164)*
#t317/t22+(t1+(t3+t153*t118)*t153)*t325/t48+(t1+(t3+t149*t118)*t149
#)*t333/t46+(-t6*t146-t21/t338)*t342+(-t6*t160-t21*t154)*t347+x*(20
#.D0*t126+60.D0*a0+30.D0*t136+15.D0*t129+12.D0*t130)/120.D0
    sensitivity_expr = t244+t357
    return
end

```

Here is the full expression for the *definite* integration result calculated in [section 4](#):

defint

$$\begin{aligned}
S(r, s, g, T, v_1, v_2) = & \frac{3}{32} \frac{a_4 \sin(2(s-r)v_2)}{(s-r)^5} + \frac{3}{32} \frac{a_4 \sin(2(s+r g)v_2)}{(s+r g)^5} - \frac{3}{32} \frac{a_4 \sin(2(s-r)v_1)}{(s-r)^5} \\
& + \frac{3}{32} \frac{a_4 \sin(2(s-r g)v_2)}{(s-r g)^5} - \frac{3}{32} \frac{a_4 \sin(2(s+r g)v_1)}{(s+r g)^5} + \frac{3}{32} \frac{a_4 \sin(2(s+r)v_2)}{(s+r)^5} \\
& - \frac{3}{32} \frac{a_4 \sin(2(s-r g)v_1)}{(s-r g)^5} + \frac{1}{2} \frac{\left(a_0 + a_1 v_1 + a_2 v_1^2 + a_3 v_1^3 + a_4 v_1^4\right) \sin((-r+r g)v_1)}{(-1+g)r}
\end{aligned}$$

$$\begin{aligned}
& -\frac{3}{32} \frac{a_4 \sin(2(s+r)v_1)}{(s+r)^5} \\
& + \left(-\frac{1}{2} \frac{2v_2 a_2 + 3v_2^2 a_3 + a_1 + 4v_2^3 a_4}{(-1+g)^2 r^2} + 3 \frac{a_3 + 4v_2 a_4}{(-1+g)^4 r^4} \right) \cos(r(-1+g)v_2) + \\
& \left(-\frac{1}{8} \frac{a_0 + a_1 v_2 + a_2 v_2^2 + a_3 v_2^3 + a_4 v_2^4}{g r} + \frac{1}{16} \frac{a_2 + 3v_2 a_3 + 6v_2^2 a_4}{g^3 r^3} - \frac{3}{16} \frac{a_4}{g^5 r^5} \right) \sin(2r g v_2) \\
& + \left(\frac{\frac{1}{8} a_3 v_1^3 + \frac{1}{8} a_4 v_1^4 + \frac{1}{8} a_1 v_1 + \frac{1}{8} a_2 v_1^2 + \frac{1}{8} a_0}{r} + \frac{-\frac{3}{16} v_1 a_3 - \frac{1}{16} a_2 - \frac{3}{8} v_1^2 a_4}{r^3} + \frac{3}{16} \frac{a_4}{r^5} \right) \\
& \sin(2r v_1) + \left(-\frac{a_2 + 3v_2 a_3 + 6v_2^2 a_4}{(1+g)^3 r^3} + 12 \frac{a_4}{(1+g)^5 r^5} \right) \sin(r(1+g)v_2) \\
& + \left(\frac{\frac{3}{16} v_1^2 a_3 + \frac{1}{8} v_1 a_2 + \frac{1}{4} v_1^3 a_4 + \frac{1}{16} a_1}{r^2} + \frac{-\frac{3}{32} a_3 - \frac{3}{8} v_1 a_4}{r^4} \right) \cos(2r v_1) \\
& + \left(\frac{a_2 + 3v_2 a_3 + 6v_2^2 a_4}{(-1+g)^3 r^3} - 12 \frac{a_4}{(-1+g)^5 r^5} \right) \sin(r(-1+g)v_2) \\
& + \left(-\frac{a_2 + 3v_1 a_3 + 6v_1^2 a_4}{(-1+g)^3 r^3} + 12 \frac{a_4}{(-1+g)^5 r^5} \right) \sin(r(-1+g)v_1) + \\
& \left(-\frac{1}{8} \frac{a_3 v_2^3 - \frac{1}{8} a_4 v_2^4 - \frac{1}{8} a_1 v_2 - \frac{1}{8} a_2 v_2^2 - \frac{1}{8} a_0}{r} + \frac{\frac{3}{16} v_2 a_3 + \frac{1}{16} a_2 + \frac{3}{8} v_2^2 a_4}{r^3} - \frac{3}{16} \frac{a_4}{r^5} \right) \\
& \sin(2r v_2) + \left(\frac{a_2 + 3v_1 a_3 + 6v_1^2 a_4}{(1+g)^3 r^3} - 12 \frac{a_4}{(1+g)^5 r^5} \right) \sin(r(1+g)v_1) \\
& + \left(-\frac{1}{16} \frac{2v_2 a_2 + 3v_2^2 a_3 + a_1 + 4v_2^3 a_4}{g^2 r^2} + \frac{3}{32} \frac{a_3 + 4v_2 a_4}{g^4 r^4} \right) \cos(2r g v_2) \\
& + \left(\frac{1}{2} \frac{2v_2 a_2 + 3v_2^2 a_3 + a_1 + 4v_2^3 a_4}{(1+g)^2 r^2} - 3 \frac{a_3 + 4v_2 a_4}{(1+g)^4 r^4} \right) \cos(r(1+g)v_2)
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{1}{8} \frac{a_0 + a_1 v_1 + a_2 v_1^2 + a_3 v_1^3 + a_4 v_1^4}{g r} - \frac{1}{16} \frac{a_2 + 3 v_1 a_3 + 6 v_1^2 a_4}{g^3 r^3} + \frac{3}{16} \frac{a_4}{g^5 r^5} \right) \sin(2 r g v_1) \\
& + \left(\frac{1}{16} \frac{2 v_1 a_2 + 3 v_1^2 a_3 + a_1 + 4 v_1^3 a_4}{g^2 r^2} - \frac{3}{32} \frac{a_3 + 4 v_1 a_4}{g^4 r^4} \right) \cos(2 r g v_1) \\
& - \frac{1}{2} \frac{\left(a_0 + a_1 v_1 + a_2 v_1^2 + a_3 v_1^3 + a_4 v_1^4 \right) \sin((r + r g) v_1)}{(1+g) r} \\
& + \frac{1}{2} \frac{\left(a_0 + a_1 v_2 + a_2 v_2^2 + a_3 v_2^3 + a_4 v_2^4 \right) \sin((r + r g) v_2)}{(1+g) r} \\
& - \frac{1}{2} \frac{\left(a_0 + a_1 v_2 + a_2 v_2^2 + a_3 v_2^3 + a_4 v_2^4 \right) \sin((-r + r g) v_2)}{(-1+g) r} \\
& + \left(\frac{-\frac{3}{16} v_2^2 a_3 - \frac{1}{8} v_2 a_2 - \frac{1}{4} v_2^3 a_4 - \frac{1}{16} a_1}{r^2} + \frac{\frac{3}{32} a_3 + \frac{3}{8} v_2 a_4}{r^4} \right) \cos(2 r v_2) + \left(\right. \\
& \left. - \frac{1}{16} \frac{v_1^3 (v_1 a_4 + a_3)}{s + r g} + \left(-\frac{1}{16} g^2 \left(a_2 v_1^2 + a_0 + a_1 v_1 \right) r^2 - \frac{1}{8} g \left(a_2 v_1^2 + a_0 + a_1 v_1 \right) s r \right. \right. \\
& \left. \left. + \left(-\frac{1}{16} a_0 - \frac{1}{16} a_2 v_1^2 - \frac{1}{16} a_1 v_1 \right) s^2 + \frac{1}{32} a_2 + \frac{3}{32} v_1 a_3 + \frac{3}{16} v_1^2 a_4 \right) / (s + r g)^3 \right) \\
& \sin((2 s + 2 r g) v_1) + \left(\frac{\frac{3}{8} v_1^2 a_3 + \frac{1}{4} v_1 a_2 + \frac{1}{2} v_1^3 a_4 + \frac{1}{8} a_1}{s^2} + \frac{-\frac{3}{16} a_3 - \frac{3}{4} v_1 a_4}{s^4} \right) \cos(2 s v_1) + \\
& \left(\frac{\frac{1}{4} a_3 v_1^3 + \frac{1}{4} a_4 v_1^4 + \frac{1}{4} a_1 v_1 + \frac{1}{4} a_2 v_1^2 + \frac{1}{4} a_0}{s} + \frac{-\frac{3}{8} v_1 a_3 - \frac{1}{8} a_2 - \frac{3}{4} v_1^2 a_4}{s^3} + \frac{3}{8} \frac{a_4}{s^5} \right) \\
& \sin(2 s v_1) + \left(\frac{-\frac{3}{8} v_2^2 a_3 - \frac{1}{4} v_2 a_2 - \frac{1}{2} v_2^3 a_4 - \frac{1}{8} a_1}{s^2} + \frac{\frac{3}{16} a_3 + \frac{3}{4} v_2 a_4}{s^4} \right) \cos(2 s v_2) + \\
& \left(\frac{-\frac{1}{4} a_3 v_2^3 - \frac{1}{4} a_4 v_2^4 - \frac{1}{4} a_1 v_2 - \frac{1}{4} a_2 v_2^2 - \frac{1}{4} a_0}{s} + \frac{\frac{3}{8} v_2 a_3 + \frac{1}{8} a_2 + \frac{3}{4} v_2^2 a_4 - \frac{3}{8} a_4}{s^3} - \frac{3}{8} \frac{a_4}{s^5} \right)
\end{aligned}$$

$$\begin{aligned}
& \left. \sin(2s v_2) + \left(-\frac{1}{2} \frac{2v_1^2 a_2 + 3v_1^2 a_3 + a_1 + 4v_1^3 a_4}{(1+g)^2 r^2} + 3 \frac{a_3 + 4v_1 a_4}{(1+g)^4 r^4} \right) \cos(r(1+g)v_1) \right. \\
& + \left. \left(\frac{1}{2} \frac{2v_1^2 a_2 + 3v_1^2 a_3 + a_1 + 4v_1^3 a_4}{(-1+g)^2 r^2} - 3 \frac{a_3 + 4v_1 a_4}{(-1+g)^4 r^4} \right) \cos(r(-1+g)v_1) + \right. \\
& - \frac{1}{4} \frac{v_2^2 (4v_2 a_4 + 3a_3)}{(2s - r - rg)^2} \\
& + \frac{-\frac{1}{4}(1+g)^2 (2v_2 a_2 + a_1) r^2 + (1+g)(2v_2 a_2 + a_1) s r + (-2v_2 a_2 - a_1) s^2 + 6v_2 a_4 + \frac{3}{2} a_3}{(2s - r - rg)^4} \\
& \left. \right. \\
& \left. \cos((2s - r - rg)v_2) + \left(-\frac{1}{4} \frac{v_2^3 (v_2 a_4 + a_3)}{2s - r - rg} + \left(-\frac{1}{4} v_2 (1+g)^2 (v_2 a_2 + a_1) r^2 \right. \right. \right. \\
& + v_2 (1+g) (v_2 a_2 + a_1) s r - v_2 (v_2 a_2 + a_1) s^2 + \frac{3}{2} v_2 (2v_2 a_4 + a_3) \Big) \Big/ (2s - r - rg)^3 + \Big(\\
& -\frac{1}{4} a_0 (1+g)^4 r^4 + 2 a_0 (1+g)^3 s r^3 + \left(-6 a_0 (1+g)^2 s^2 + \frac{1}{2} a_2 (1+g)^2 \right) r^2 \\
& \left. \left. \left. + (8 a_0 (1+g) s^3 - 2 a_2 (1+g) s) r - 4 a_0 s^4 - 6 a_4 + 2 a_2 s^2 \right) \Big/ (2s - r - rg)^5 \right) \right. \\
& \left. \sin((2s - r - rg)v_2) + \left(-\frac{1}{32} \frac{v_1^2 (4v_1 a_4 + 3a_3)}{(s + rg)^2} \right. \right. \\
& + \frac{-\frac{1}{32} g^2 (2v_1 a_2 + a_1) r^2 - \frac{1}{16} g (2v_1 a_2 + a_1) s r + \left(-\frac{1}{16} v_1 a_2 - \frac{1}{32} a_1 \right) s^2 + \frac{3}{16} v_1 a_4 + \frac{3}{64} a_3}{(s + rg)^4} \\
& \left. \right. \\
& \left. \right. \\
& \left. \cos((2s + 2rg)v_1) + \left(\frac{1}{16} \frac{v_2^3 (v_2 a_4 + a_3)}{s + rg} + \left(\frac{1}{16} g^2 \left(a_2 v_2^2 + a_0 + a_1 v_2 \right) r^2 \right. \right. \right. \\
& + \frac{1}{8} g \left(a_2 v_2^2 + a_0 + a_1 v_2 \right) s r + \left(\frac{1}{16} a_0 + \frac{1}{16} a_2 v_2^2 + \frac{1}{16} a_1 v_2 \right) s^2 - \frac{3}{32} v_2 a_3 - \frac{1}{32} a_2 - \frac{3}{16} v_2^2 a_4 \\
& \left. \left. \left. \right) \Big/ (s + rg)^3 \right) \sin((2s + 2rg)v_2) + \left(\frac{1}{4} \frac{v_1^2 (4v_1 a_4 + 3a_3)}{(2s - r - rg)^2} \right. \right. \\
\end{aligned}$$

$$\begin{aligned}
& + \frac{\frac{1}{4}(1+g)^2(2v_1a_2+a_1)r^2 - (1+g)(2v_1a_2+a_1)s r + (2v_1a_2+a_1)s^2 - 6v_1a_4 - \frac{3}{2}a_3}{(2s-r-r g)^4} \\
& \cos((2s-r-r g)v_1) + \left(\frac{1}{4} \frac{v_1^3(v_1a_4+a_3)}{2s-r-r g} + \left(\frac{1}{4}v_1(1+g)^2(v_1a_2+a_1)r^2 \right. \right. \\
& \left. \left. - v_1(1+g)(v_1a_2+a_1)s r + (v_1a_2+a_1)v_1s^2 - \frac{3}{2}v_1(2v_1a_4+a_3) \right) \right/ (2s-r-r g)^3 + \left(\right. \\
& \frac{1}{4}a_0(1+g)^4r^4 - 2a_0(1+g)^3s r^3 + \left(6a_0(1+g)^2s^2 - \frac{1}{2}a_2(1+g)^2 \right) r^2 \\
& \left. + (-8a_0(1+g)s^3 + 2a_2(1+g)s)r + 4a_0s^4 + 6a_4 - 2a_2s^2 \right) \right/ (2s-r-r g)^5 \\
& \sin((2s-r-r g)v_1) + \left(-\frac{1}{4} \frac{v_1^3(v_1a_4+a_3)}{2s-r+r g} + \left(-\frac{1}{4}v_1(-1+g)^2(v_1a_2+a_1)r^2 \right. \right. \\
& \left. \left. - v_1(-1+g)(v_1a_2+a_1)s r - (v_1a_2+a_1)v_1s^2 + \frac{3}{2}v_1(2v_1a_4+a_3) \right) \right) \right/ (2s-r+r g)^3 + \left(\right. \\
& - \frac{1}{4}a_0(-1+g)^4r^4 - 2a_0(-1+g)^3s r^3 + \left(-6a_0(-1+g)^2s^2 + \frac{1}{2}a_2(-1+g)^2 \right) r^2 \\
& \left. + (-8a_0(-1+g)s^3 + 2a_2(-1+g)s)r - 4a_0s^4 - 6a_4 + 2a_2s^2 \right) \right) \right/ (2s-r+r g)^5 \\
& \sin((2s-r+r g)v_1) + \left(\frac{1}{4} \frac{v_2^2(4v_2a_4+3a_3)}{(2s-r+r g)^2} \right. \\
& \left. + \frac{\frac{1}{4}(-1+g)^2(2v_2a_2+a_1)r^2 + (-1+g)(2v_2a_2+a_1)s r + (2v_2a_2+a_1)s^2 - 6v_2a_4 - \frac{3}{2}a_3}{(2s-r+r g)^4} \right) \\
& \cos((2s-r+r g)v_2) + \left(\frac{1}{4} \frac{v_2^2(4v_2a_4+3a_3)}{(2s+r-r g)^2} \right. \\
& \left. + \frac{\frac{1}{4}(-1+g)^2(2v_2a_2+a_1)r^2 - (-1+g)(2v_2a_2+a_1)s r + (2v_2a_2+a_1)s^2 - 6v_2a_4 - \frac{3}{2}a_3}{(2s+r-r g)^4} \right)
\end{aligned}$$

$$\begin{aligned}
& \cos((2s+r-rg)v_2) + \left(-\frac{1}{4} \frac{v_1^2 (4v_1 a_4 + 3a_3)}{(2s-r+rg)^2} + \right. \\
& \left. -\frac{1}{4} (-1+g)^2 (2v_1 a_2 + a_1) r^2 - (-1+g) (2v_1 a_2 + a_1) s r + (-2v_1 a_2 - a_1) s^2 + 6v_1 a_4 + \frac{3}{2} a_3 \right. \\
& \left. \frac{(2s-r+rg)^4}{(2s-r+rg)^4} \right) \\
& \cos((2s-r+rg)v_1) + \left(\frac{1}{4} \frac{v_1^2 (4v_1 a_4 + 3a_3)}{(2s+r+rg)^2} \right. \\
& \left. + \frac{\frac{1}{4} (1+g)^2 (2v_1 a_2 + a_1) r^2 + (1+g) (2v_1 a_2 + a_1) s r + (2v_1 a_2 + a_1) s^2 - 6v_1 a_4 - \frac{3}{2} a_3}{(2s+r+rg)^4} \right) \\
& \cos((2s+r+rg)v_1) + \left(-\frac{1}{4} \frac{v_1^3 (v_1 a_4 + a_3)}{2s+r-rg} + \left(-\frac{1}{4} v_1 (-1+g)^2 (v_1 a_2 + a_1) r^2 \right. \right. \\
& \left. + v_1 (-1+g) (v_1 a_2 + a_1) s r - (v_1 a_2 + a_1) v_1 s^2 + \frac{3}{2} v_1 (2v_1 a_4 + a_3) \right) \Big/ (2s+r-rg)^3 + \left(\right. \\
& \left. -\frac{1}{4} a_0 (-1+g)^4 r^4 + 2 a_0 (-1+g)^3 s r^3 + \left(-6 a_0 (-1+g)^2 s^2 + \frac{1}{2} a_2 (-1+g)^2 \right) r^2 \right. \\
& \left. + (8 a_0 (-1+g) s^3 - 2 a_2 (-1+g) s) r - 4 a_0 s^4 - 6 a_4 + 2 a_2 s^2 \right) \Big/ (2s+r-rg)^5 \Bigg) \\
& \sin((2s+r-rg)v_1) + \left(\frac{1}{32} \frac{v_2^2 (4v_2 a_4 + 3a_3)}{(s+r)^2} \right. \\
& \left. + \left(\frac{1}{16} v_2 a_2 + \frac{1}{32} a_1 \right) r^2 + \left(\frac{1}{8} v_2 a_2 + \frac{1}{16} a_1 \right) s r + \left(\frac{1}{16} v_2 a_2 + \frac{1}{32} a_1 \right) s^2 - \frac{3}{16} v_2 a_4 - \frac{3}{64} a_3 \right. \\
& \left. \frac{(s+r)^4}{(s+r)^4} \right) \\
& \cos((2s+2r)v_2) + \left(-\frac{1}{16} \frac{v_1^3 (v_1 a_4 + a_3)}{s+r} + \left(\left(-\frac{1}{16} a_0 - \frac{1}{16} a_2 v_1^2 - \frac{1}{16} a_1 v_1 \right) r^2 \right. \right. \\
& \left. + \left(-\frac{1}{8} a_0 - \frac{1}{8} a_2 v_1^2 - \frac{1}{8} a_1 v_1 \right) s r + \left(-\frac{1}{16} a_0 - \frac{1}{16} a_2 v_1^2 - \frac{1}{16} a_1 v_1 \right) s^2 + \frac{3}{32} v_1 a_3 + \frac{1}{32} a_2 \right. \\
& \left. + \frac{3}{16} v_1^2 a_4 \right) \Big/ (s+r)^3 \Bigg) \\
& \sin((2s+2r)v_1) + \left(\frac{1}{16} \frac{v_2^3 (v_2 a_4 + a_3)}{s+r} + \left(\right. \right. \\
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{1}{16} a_0 + \frac{1}{16} a_2 v_2^2 + \frac{1}{16} a_1 v_2 \right) r^2 + \left(\frac{1}{8} a_0 + \frac{1}{8} a_2 v_2^2 + \frac{1}{8} a_1 v_2 \right) s r \\
& + \left(\frac{1}{16} a_0 + \frac{1}{16} a_2 v_2^2 + \frac{1}{16} a_1 v_2 \right) s^2 - \frac{3}{32} v_2 a_3 - \frac{1}{32} a_2 - \frac{3}{16} v_2^2 a_4 \Bigg) / (s+r)^3 \Bigg) \\
& \sin((2s+2r)v_2) + \left(-\frac{1}{4} \frac{v_2^2 (4v_2 a_4 + 3a_3)}{(2s+r+rg)^2} \right. \\
& + \frac{-\frac{1}{4}(1+g)^2 (2v_2 a_2 + a_1) r^2 - (1+g)(2v_2 a_2 + a_1) s r + (-2v_2 a_2 - a_1) s^2 + 6v_2 a_4 + \frac{3}{2} a_3}{(2s+r+rg)^4} \\
& \Bigg) \cos((2s+r+rg)v_2) + \left(\frac{1}{4} \frac{v_2^3 (v_2 a_4 + a_3)}{2s+r-rg} + \left(\frac{1}{4} v_2 (-1+g)^2 (v_2 a_2 + a_1) r^2 \right. \right. \\
& - v_2 (-1+g) (v_2 a_2 + a_1) s r + v_2 (v_2 a_2 + a_1) s^2 - \frac{3}{2} v_2 (2v_2 a_4 + a_3) \Bigg) / (2s+r-rg)^3 + \left(\right. \\
& \frac{1}{4} a_0 (-1+g)^4 r^4 - 2 a_0 (-1+g)^3 s r^3 + \left(6 a_0 (-1+g)^2 s^2 - \frac{1}{2} a_2 (-1+g)^2 \right) r^2 \\
& \left. \left. + (-8 a_0 (-1+g) s^3 + 2 a_2 (-1+g) s) r + 4 a_0 s^4 + 6 a_4 - 2 a_2 s^2 \right) \right) / (2s+r-rg)^5 \Bigg) \\
& \sin((2s+r-rg)v_2) + \left(-\frac{1}{4} \frac{v_2^3 (v_2 a_4 + a_3)}{2s+r+rg} + \left(-\frac{1}{4} v_2 (1+g)^2 (v_2 a_2 + a_1) r^2 \right. \right. \\
& - v_2 (1+g) (v_2 a_2 + a_1) s r - v_2 (v_2 a_2 + a_1) s^2 + \frac{3}{2} v_2 (2v_2 a_4 + a_3) \Bigg) / (2s+r+rg)^3 + \left(\right. \\
& - \frac{1}{4} a_0 (1+g)^4 r^4 - 2 a_0 (1+g)^3 s r^3 + \left(-6 a_0 (1+g)^2 s^2 + \frac{1}{2} a_2 (1+g)^2 \right) r^2 \\
& \left. \left. + (-8 a_0 (1+g) s^3 + 2 a_2 (1+g) s) r - 4 a_0 s^4 - 6 a_4 + 2 a_2 s^2 \right) \right) / (2s+r+rg)^5 \Bigg) \\
& \sin((2s+r+rg)v_2) + \left(\frac{1}{4} \frac{v_1^3 (v_1 a_4 + a_3)}{2s+r+rg} + \left(\frac{1}{4} v_1 (1+g)^2 (v_1 a_2 + a_1) r^2 \right. \right. \\
& + v_1 (1+g) (v_1 a_2 + a_1) s r + (v_1 a_2 + a_1) v_1 s^2 - \frac{3}{2} v_1 (2v_1 a_4 + a_3) \Bigg) / (2s+r+rg)^3 + \left(\right. \\
& \frac{1}{4} a_0 (1+g)^4 r^4 + 2 a_0 (1+g)^3 s r^3 + \left(6 a_0 (1+g)^2 s^2 - \frac{1}{2} a_2 (1+g)^2 \right) r^2
\end{aligned}$$

$$\begin{aligned}
& + (8 a_0 (1+g) s^3 - 2 a_2 (1+g) s) r + 4 a_0 s^4 + 6 a_4 - 2 a_2 s^2 \Big) \Big/ (2 s + r + r g)^5 \\
& \sin((2 s + r + r g) v_1) + \left(\frac{1}{32} \frac{v_2^2 (4 v_2 a_4 + 3 a_3)}{(s-r)^2} \right. \\
& + \frac{\left(\frac{1}{16} v_2 a_2 + \frac{1}{32} a_1 \right) r^2 + \left(-\frac{1}{8} v_2 a_2 - \frac{1}{16} a_1 \right) s r + \left(\frac{1}{16} v_2 a_2 + \frac{1}{32} a_1 \right) s^2 - \frac{3}{16} v_2 a_4 - \frac{3}{64} a_3}{(s-r)^4} \Big) \\
& \cos((2 s - 2 r) v_2) + \left(-\frac{1}{16} \frac{v_1^3 (v_1 a_4 + a_3)}{s-r} + \left(\left(-\frac{1}{16} a_0 - \frac{1}{16} a_2 v_1^2 - \frac{1}{16} a_1 v_1 \right) r^2 \right. \right. \\
& + \left(\frac{1}{8} a_0 + \frac{1}{8} a_2 v_1^2 + \frac{1}{8} a_1 v_1 \right) s r + \left(-\frac{1}{16} a_0 - \frac{1}{16} a_2 v_1^2 - \frac{1}{16} a_1 v_1 \right) s^2 + \frac{3}{32} v_1 a_3 + \frac{1}{32} a_2 \right. \\
& \left. \left. + \frac{3}{16} v_1^2 a_4 \right) \Big/ (s-r)^3 \right) \sin((2 s - 2 r) v_1) + \left(\frac{1}{16} \frac{v_2^3 (v_2 a_4 + a_3)}{s-r} + \left(\right. \right. \\
& \left. \left. \left(\frac{1}{16} a_0 + \frac{1}{16} a_2 v_2^2 + \frac{1}{16} a_1 v_2 \right) r^2 + \left(-\frac{1}{8} a_0 - \frac{1}{8} a_2 v_2^2 - \frac{1}{8} a_1 v_2 \right) s r \right. \right. \\
& \left. \left. + \left(\frac{1}{16} a_0 + \frac{1}{16} a_2 v_2^2 + \frac{1}{16} a_1 v_2 \right) s^2 - \frac{3}{32} v_2 a_3 - \frac{1}{32} a_2 - \frac{3}{16} v_2^2 a_4 \right) \Big/ (s-r)^3 \right) \\
& \sin((2 s - 2 r) v_2) + \left(-\frac{1}{4} \frac{v_1^2 (4 v_1 a_4 + 3 a_3)}{(2 s + r - r g)^2} + \right. \\
& \left. \left. - \frac{1}{4} (-1+g)^2 (2 v_1 a_2 + a_1) r^2 + (-1+g) (2 v_1 a_2 + a_1) s r + (-2 v_1 a_2 - a_1) s^2 + 6 v_1 a_4 + \frac{3}{2} a_3 \right. \right. \\
& \left. \left. \right. \right. \\
& \left. \left. \left. \left. \cos((2 s + r - r g) v_1) + \frac{1}{120} (-v_1 + v_2) \left(12 a_4 v_2^4 + 15 a_3 v_2^3 + 12 v_2^3 a_4 v_1 + 20 a_2 v_2^2 \right. \right. \right. \right. \\
& \left. \left. \left. \left. + 15 v_2^2 v_1 a_3 + 12 v_2^2 a_4 v_1^2 + 30 a_1 v_2 + 20 v_2 v_1 a_2 + 15 v_2 v_1^2 a_3 + 12 v_2 v_1^3 a_4 + 60 a_0 \right. \right. \right. \right. \\
& \left. \left. \left. \left. + 30 a_1 v_1 + 20 a_2 v_1^2 + 15 a_3 v_1^3 + 12 a_4 v_1^4 \right) + \left(-\frac{1}{32} \frac{v_1^2 (4 v_1 a_4 + 3 a_3)}{(s+r)^2} \right. \right. \right. \right. \\
\end{aligned}$$

$$\begin{aligned}
& + \frac{\left(-\frac{1}{16} v_1 a_2 - \frac{1}{32} a_1 \right) r^2 + \left(-\frac{1}{8} v_1 a_2 - \frac{1}{16} a_1 \right) s r + \left(-\frac{1}{16} v_1 a_2 - \frac{1}{32} a_1 \right) s^2 + \frac{3}{16} v_1 a_4 + \frac{3}{64} a_3}{(s+r)^4} \\
& \cos((2s+2r)v_1) + \left(-\frac{1}{32} \frac{v_1^2 (4v_1 a_4 + 3a_3)}{(s-r)^2} \right. \\
& + \frac{\left(-\frac{1}{16} v_1 a_2 - \frac{1}{32} a_1 \right) r^2 + \left(\frac{1}{8} v_1 a_2 + \frac{1}{16} a_1 \right) s r + \left(-\frac{1}{16} v_1 a_2 - \frac{1}{32} a_1 \right) s^2 + \frac{3}{16} v_1 a_4 + \frac{3}{64} a_3}{(s-r)^4} \\
& \cos((2s-2r)v_1) + \left(\frac{1}{4} \frac{v_2^3 (v_2 a_4 + a_3)}{2s-r+rg} + \left(\frac{1}{4} v_2 (-1+g)^2 (v_2 a_2 + a_1) r^2 \right. \right. \\
& + v_2 (-1+g) (v_2 a_2 + a_1) s r + v_2 (v_2 a_2 + a_1) s^2 - \frac{3}{2} v_2 (2v_2 a_4 + a_3) \Big) \Big/ (2s-r+rg)^3 + \left(\right. \\
& \frac{1}{4} a_0 (-1+g)^4 r^4 + 2 a_0 (-1+g)^3 s r^3 + \left(6 a_0 (-1+g)^2 s^2 - \frac{1}{2} a_2 (-1+g)^2 \right) r^2 \\
& \left. \left. + (8 a_0 (-1+g) s^3 - 2 a_2 (-1+g) s) r + 4 a_0 s^4 + 6 a_4 - 2 a_2 s^2 \right) \Big/ (2s-r+rg)^5 \right) \\
& \sin((2s-r+rg)v_2) + \left(\frac{1}{16} \frac{v_2^3 (v_2 a_4 + a_3)}{s-r g} + \left(\frac{1}{16} g^2 \left(a_2 v_2^2 + a_0 + a_1 v_2 \right) r^2 \right. \right. \\
& - \frac{1}{8} g \left(a_2 v_2^2 + a_0 + a_1 v_2 \right) s r + \left(\frac{1}{16} a_0 + \frac{1}{16} a_2 v_2^2 + \frac{1}{16} a_1 v_2 \right) s^2 - \frac{3}{32} v_2 a_3 - \frac{1}{32} a_2 - \frac{3}{16} v_2^2 a_4 \\
& \left. \left. \right) \Big/ (s-r g)^3 \right) \sin((2s-2rg)v_2) + \left(\frac{1}{32} \frac{v_2^2 (4v_2 a_4 + 3a_3)}{(s+rg)^2} \right. \\
& + \frac{\frac{1}{32} g^2 (2v_2 a_2 + a_1) r^2 + \frac{1}{16} g (2v_2 a_2 + a_1) s r + \left(\frac{1}{16} v_2 a_2 + \frac{1}{32} a_1 \right) s^2 - \frac{3}{16} v_2 a_4 - \frac{3}{64} a_3}{(s+rg)^4} \\
& \cos((2s+2rg)v_2) + \left(-\frac{1}{32} \frac{v_1^2 (4v_1 a_4 + 3a_3)}{(s-r g)^2} \right. \\
& + \frac{-\frac{1}{32} g^2 (2v_1 a_2 + a_1) r^2 + \frac{1}{16} g (2v_1 a_2 + a_1) s r + \left(-\frac{1}{16} v_1 a_2 - \frac{1}{32} a_1 \right) s^2 + \frac{3}{16} v_1 a_4 + \frac{3}{64} a_3}{(s-r g)^4}
\end{aligned}$$

$$\begin{aligned}
& \left. \left(\cos((2s - 2rg)v_1) + \left(\frac{1}{32} \frac{v_2^2 (4v_2 a_4 + 3a_3)}{(s - rg)^2} \right. \right. \right. \\
& \left. \left. \left. + \frac{\frac{1}{32} g^2 (2v_2 a_2 + a_1) r^2 - \frac{1}{16} g (2v_2 a_2 + a_1) s r + \left(\frac{1}{16} v_2 a_2 + \frac{1}{32} a_1 \right) s^2 - \frac{3}{16} v_2 a_4 - \frac{3}{64} a_3}{(s - rg)^4} \right) \right. \\
& \left. \left. \left. \cos((2s - 2rg)v_2) + \left(-\frac{1}{16} \frac{v_1^3 (v_1 a_4 + a_3)}{s - rg} + \left(-\frac{1}{16} g^2 \left(a_2 v_1^2 + a_0 + a_1 v_1 \right) r^2 \right. \right. \right. \right. \\
& \left. \left. \left. \left. + \frac{1}{8} g \left(a_2 v_1^2 + a_0 + a_1 v_1 \right) s r + \left(-\frac{1}{16} a_0 - \frac{1}{16} a_2 v_1^2 - \frac{1}{16} a_1 v_1 \right) s^2 + \frac{1}{32} a_2 + \frac{3}{32} v_1 a_3 \right. \right. \right. \right. \\
& \left. \left. \left. \left. + \frac{3}{16} v_1^2 a_4 \right) \right) \right) \right) \right) \sin((2s - 2rg)v_1)
\end{aligned}$$

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[ save integrand, indef, cleanup, sensitivity_expr, integral, ugh, pieces, last_integral, parts, defint, denoms,
blowups, get_denoms, "d:/FAMEStuff/PhotonSensitivity/SensitivityIntegral.eqn"
[ save "d:/FAMEStuff/PhotonSensitivity/SensitivityIntegral.m"
[ ?

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